NULL-ADDITIVITY IN THE THEORY OF ALGORITHMIC RANDOMNESS

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Abstract. In this paper, we develop a general framework integrating algorithmic and higher randomness theories. We clarify the relationship of the notions of *triviality* and *uniform-lowness* in algorithmic randomness theory and *null-additivity* in set theory by effectivizing combinatorial characterizations of transitive additivity in set theory of the real line. For instance, we show that the following three conditions are equivalent for an infinite binary sequence A: (1) A is low for Kurtz randomness with respect to uniform relativization; (2) $\{A\}$ is effectively $\mathcal{E}$-additive; (3) $A \triangle Z$ is Kurtz random whenever Z is Kurtz random. Additionally, we study levels of uniformity associated with lowness for randomness and additivity numbers over various levels of algorithmic/higher randomness theories. We also clarify the relationship between the Kučera-Gács Theorem and strong measure zero sets of reals over Spector pointclasses, and we show an abstract version of the Kučera-Gács Theorem stating that for every Spector pointclass Γ, every real is Γ-weak-truth-reducible to a Γ-Martin-Löf random real.

1. Introduction

1.1. **Historical Background.** In the history of the development of computability theory and set theory, they are influenced by each other, and thus these two theories share many common notions and techniques. For instance, the forcing method is one of the most important techniques common to both of these theories, and it involves another important common notion, so-called *randomness*. One way of formalizing randomness is to define it as genericity with respect to (idealized forcing [3, 38, 78] obtained from) the σ-ideal of Lebesgue null sets, that is, a point x avoiding all Lebegue null sets whose Borel codes are contained in a ground model $\mathbf{V}$, or equivalently, the singleton $\{x\}$ is not Lebesgue null over $\mathbf{V}$. Here, in our paper, $\mathbf{V}$ can be chosen as a model of ZFC set theory, a Turing ideal (i.e., an ω-model of RCA, see [70]), or an admissible set (i.e., a model of KP, see [4]).

Generally, (quasi-)genericity is defined not only for the σ-ideal of Lebesgue null sets, but also for any ideal $\mathcal{I}$ (on a Polish space) such as σ-ideals of meager sets, sets of Hausdorff dimension zero, etc. Here, a point x is quasi-generic with respect to an ideal $\mathcal{I}$ if the singleton $\{x\}$ is not covered by a $\mathcal{I}$-negligible set coded in a ground model. In computability theory, it is known that the Hausdorff dimension of the singleton $\{x\}$ (in an outer model) over the ground model consisting of all computable sets corresponds to the growing rate of the prefix-free Kolmogorov complexity (see [24, Section 13]). We also have similar Kolmogorov complexity characterizations of the hierarchy of Hausdorff outer measures and packing outer measures with respect to gauge functions (see also [62]). Intuitively, this suggests that the level of non-randomness of an individual point coincides with the level of capturability of an individual point by effectively-negligible sets in a measure-theoretic sense. In fact, these results produce the relationship between computational *lowness* and *traceability*.

The technical notion called *traceability* (also called *slalom*; see [13, 63, 64]) is also shared by set theorists and computability theorists. Indeed, the notion of traceability in computability theory was first introduced, inspired by Raisonnier's proof in set theory, by Terwijn and Zambella [74] to characterize lowness for Schnorr randomness (see [24, 57]). Another important notion in the theory of algorithmic randomness is K-triviality, introduced by Chaitin [18] as opposite of incompressibility, and has been studied by Solovay [71] and others (see [57]). One of the most surprising achievements in this field is the equivalence between K-triviality and lowness for Martin-Löf randomness (see [56, 35]). Later, it is also shown that Schnorr triviality [23] is equivalent to the *truth-table* version of lowness for Schnorr randomness [28]. The first, to define lowness, uses the usual oracle (Turing) relativization, and the second uses *uniform relativization* (Miyabe [52], Miyabe-Rute [51]).

In this context, the notion of lowness comparing two randomness notions is also important. For any randomness notions $\mathcal{R}$ and $\mathcal{S}$, a set A is low for $(\mathcal{R}, \mathcal{S})$ (written as $A \in \mathrm{Low}(\mathcal{R}, \mathcal{S})$, see [27, 45]) if every $\mathcal{R}$-random set is $\mathcal{S}$-random relative to A. With this background, Kihara-Miyabe [44] have studied two levels of lowness notions $\mathrm{Low}(\mathcal{R}, \mathcal{S})$ and $\mathrm{Low}^\star(\mathcal{R}, \mathcal{S})$ inspired by the cardinal characteristics $\mathsf{add}(\mathcal{I}, \mathcal{J})$ and $\mathsf{add}^\star(\mathcal{I}, \mathcal{J})$ in set theory, and characterize them via variants of traceability and complexity. Here, $A \in \mathrm{Low}^\star(\mathcal{R}, \mathcal{S})$ if every $\mathcal{R}$-random set is $\mathcal{S}$-random *uniformly* relative to A.

In set theory, given σ-ideals $\mathcal{I}$ and $\mathcal{J}$ on 2^ω, we say that a set $X \subseteq 2^\omega$ is $(\mathcal{I}, \mathcal{J})$-*additive* (written as $X \in \mathrm{Add}^\star(\mathcal{I}, \mathcal{J})$) if $X + I \in \mathcal{J}$ whenever $I \in \mathcal{I}$, where $X + I = \{A \triangle B : A \in X \text{ and } B \in I\}$ (see also Definition 2.36). The following four transitive additivity notions are known to have slalom (traceability) characterizations:

- $X \subseteq 2^\omega$ is *strong measure zero* if $(\mathcal{E}, \mathcal{N})$-additive.
- $X \subseteq 2^\omega$ is *meager-additive* if $(\mathcal{M}, \mathcal{M})$-additive.
- $X \subseteq 2^\omega$ is $\mathcal{E}$-*additive* if $(\mathcal{E}, \mathcal{E})$-additive.
- $X \subseteq 2^\omega$ is *null-additive* if $(\mathcal{N}, \mathcal{N})$-additive.

Here, $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{E}$ are the σ-ideals generated by meager sets, null sets, and closed null sets, respectively. The following implications are known (see [3]):

$$\text{strongly measure zero} \Rightarrow \text{meager-additive} \Leftrightarrow \mathcal{E}\text{-additive} \Rightarrow \text{null-additive}$$

This is a uniform version of the *additivity number* $\mathsf{add}(\mathcal{I}, \mathcal{J})$. We say that a set $X \subseteq 2^\omega$ is *small for* $(\mathcal{I}, \mathcal{J})$ (written as $X \in \mathrm{Add}(\mathcal{I}, \mathcal{J})$) if $\bigcup_{x \in X} N_x \in \mathcal{J}$ for every X-indexed family $\{N_x\}_{x \in X} \subseteq \mathcal{I}$. Obviously, every $(\mathcal{I}, \mathcal{J})$-small set is $(\mathcal{I}, \mathcal{J})$-additive. Although $(\mathcal{I}, \mathcal{J})$-smallness only depends on the cardinality, it will turn out to be useful when we introduce uniformity levels between $\mathrm{Add}^\star(\mathcal{I}, \mathcal{J})$ and $\mathrm{Add}(\mathcal{I}, \mathcal{J})$. For instance, Recław [61] introduced the notion of $\mathsf{add}(\mathcal{N})$-smallness as a continuous-uniform version of smallness for $\mathsf{add}(\mathcal{N})$. Moreover, Bartoszyński-Judah [2] introduced the notion of an $\mathrm{R}^\mathcal{J}$ set and an $\mathrm{SR}^\mathcal{J}$ set as Borel-uniform smallness for cardinal characteristics $\mathsf{cov}(\mathcal{J})$ and $\mathsf{add}(\mathcal{J})$, respectively. More generally, Pawlikowski-Recław [60] introduced continuous and Borel-uniform smallness for any element of Cichoń's diagram (see also [12, 68]).

Computability theoretic aspects of cardinal characteristics in Cichoń's diagram have been studied by Rupprecht [63, 64] and Brendle et al. [13]. As an earlier use of transitive additivity notions in computability theory, Higuchi-Kihara [32] pointed out the importance of the effectivization of strong measure zero $\mathrm{Add}^\star(\mathcal{E}, \mathcal{N})$ in the study of the Muchnik degree structure (indeed, diminutiveness, smallness, and very smallness in the sense of Binns [7, 8] are regarded as effectivizations of strong measure zero, the property (T') [58, 79], and null-additivity, respectively). Later, Kihara-Miyabe [44] used the notion of effective strong measure zero to give a traceability characterization of the notion of $\mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$. Other results on transitive additivity in algorithmic randomness theory can also be found in [21].

There are also various works on randomness at intermediate levels between computability theory and set theory. See [48, 39, 66, 73] for hyperarithmetical randomness, and [57, Chapter 9] for lowness and traceability in the context of higher randomness theory. Carl and Schlicht [17] also studied measure-theoretic aspects of infinite time register machine computability and infinite time Turing machine computability. In such a *higher* randomness theory, we have various uniformity levels other than uniform and non-uniform. For instance, Bienvenu-Greenberg-Monin [5] pointed out the importance of *continuous-uniform relativization* to study lowness for Π^1_1-Martin-Löf randomness.

It is straightforward to see that for a real x, its singleton $\{x\}$ is *small for* $\mathcal{N}$ *over a ground model* $\mathbf{V}$ (i.e., $\{x\} \in \mathrm{Add}(\mathcal{N})^\mathbf{V}$) if and only if x is *low for randomness-tests over* $\mathbf{V}$ (see also Proposition 3.9 for more details). We will also see that the uniform lowness notions $\mathrm{Low}^\star(\mathcal{R}, \mathcal{S})$ are related to the effectivizations of transitive additivity notions $\mathrm{Add}^\star(\mathcal{I}, \mathcal{J})$. Then, it is natural to study intermediate uniformity levels of lowness and additivity. These observations lead us to the general framework unifying various developments on *lowness for* $\mathcal{I}$-*randomness tests w.r.t.* $\mathcal{F}$-*uniform relativization over* $\mathbf{V}$ (or simply say $(\mathcal{I}; \mathcal{F}, \mathbf{V})$-lowness), where $\mathcal{I}$ is a σ-ideal and $\mathcal{F}$ is a level of uniformity (e.g., continuity, Borel measurability, etc.)

The reader may feel that our framework is quite involved, but we believe that this is an important task under the current situation of the theory of higher (infinitary) computability and randomness. In recent years, the basic computability/randomness theory for each special model of infinitary computation has been repeatedly reproduced in an ad-hoc manner. This is the reason why we wish to emphasize a general framework of generalized recursion theory which integrates various kinds of infinitary computations, even though we think that our main results are still meaningful in the algorithmic (i.e., finite-time) randomness theory.

1.2. **Summary.** Our goal of this paper is to shed light on set-theoretic structures buried in algorithmic randomness theory. In Section 2, we propose a general framework integrating algorithmic randomness theory and various kinds of higher randomness theories. In Section 3, we point out several equivalences among notions from algorithmic randomness theory and those from set theory. In Section 4, we effectivize combinatorial characterizations of transitive additivity in set theory of the real line, and clarify the relationship among the notions of *triviality* and *uniform lowness* in algorithmic randomness theory and *null-additivity* in set theory. For instance, our results imply the following equivalences in algorithmic randomness theory.

Theorem 1.1. *We have the following equivalences, where symbols A and Z range over subsets of ω.*

(i) *A is low for Schnorr randomness with respect to uniform relativization (i.e., $A \in \mathrm{Low}^\star(\mathrm{SR})$) if and only if $\{A\}$ is Schnorr null-additive, that is, $\{A\} + N := \{A \triangle Z : Z \in N\}$ is Schnorr null whenever $N \subseteq 2^\omega$ is Schnorr null.*

(ii) *A is low for (Martin-Löf,Schnorr)-randomness with respect to uniform relativization (i.e., $A \in \mathrm{Low}^\star(\mathrm{MLR},\mathrm{SR})$) if and only if A is (Martin-Löf,Schnorr)-randomness preserving, that is, $A \triangle Z$ is Schnorr random whenever Z is ML random.*

(iii) *A is low for (Martin-Löf,Kurtz)-randomness with respect to uniform relativization (i.e., $A \in \mathrm{Low}^\star(\mathrm{MLR},\mathrm{WR})$) if and only if A is (Martin-Löf,Kurtz)-randomness preserving, that is, $A \triangle Z$ is Kurtz random whenever Z is ML random.*

(iv) *A is low for Kurtz randomness with respect to uniform relativization (i.e., $A \in \mathrm{Low}^\star(\mathrm{WR})$) if and only if A is Kurtz randomness preserving, that is, $A \triangle Z$ is Kurtz random whenever Z is Kurtz random.*

(v) *A is low for weak 1-genericity with respect to uniform relativization (i.e., $A \in \mathrm{Low}^\star(\mathrm{W1G})$) if and only if A is weak-1-genericity preserving, that is, $A \triangle Z$ is weakly 1-generic whenever Z is weakly 1-generic.*

More generally, we show similar results in higher randomness theory, infinite time Turing machine randomness theory, etc; e.g., lowness for Δ^1_1 randomness with respect to uniform relativization is equivalent to Δ^1_1-null-additivity. Additionally, in Section 5, we study levels of uniformity associated with lowness for randomness and additivity numbers, where the notions of uniform lowness and transitive additivity lie at the strongest uniformity level. As a corollary, we generalize the equivalence of lowness and traceability to any uniformity level in various kinds of higher randomness theories (e.g., infinite time Turing/register machine randomness theory). In Section 6, we reveal the relationship between the Kučera-Gács Theorem and strong measure zero sets of reals over certain models. We show in Theorem 6.3 that for any Spector pointclass Γ, a Γ-semicoded closed set P is Δ-strong measure zero if and only if every real is $\mathrm{wtt}(\Gamma)$-reducible to an element of P. As a corollary, we show an abstract version of Kučera-Gács Theorem 6.1 stating that every real is $\mathrm{wtt}(\Gamma)$-reducible to a Γ-Martin-Löf random real. Finally, in Section 6.2, we see basic properties of higher versions of K-triviality and Schnorr triviality.

2. Preliminaries

We refer to [24, 57] for the background in algorithmic randomness and to the textbooks [4, 25, 34, 67] for generalized recursion theory (computability theory beyond hyperarithmetic).

2.1. Codes and Models.

2.1.1. *$\mathcal{O}\omega$-Represented Sets.* Most mathematical objects which we focus on will be coded by ω^ω or $\mathcal{P}\omega$. For instance, in set theory, we frequently identify a Borel set in an underlying space with a so-called *Borel code* (Example 2.3; see also [10, 38]). The notion of Borel coding is very useful in our study since we want to treat effectivization/relativization of null sets and meager sets in a unified way. Indeed, in algorithmic randomness theory, a Martin-Löf null set can be identified with a null set which has a nice c.e. $\mathcal{P}\omega$-name w.r.t. Borel coding in a certain sense. If $\mathcal{I}$ is a Borel σ-ideal, we think of Borel coding as a *multi-representation* [75] of $\mathcal{I}$ in a manner that every Borel code of an $\mathcal{I}$-negligible set $N \in \mathcal{I}$ is also a name of any smaller set $M \subseteq N$.

For instance, if N is included in a Lebesgue null set which has a c.e. Borel code, we shall say that a set N is c.e. null even when N itself has no exact c.e. description.

In this paper, we often use a (non-identical) $\mathcal{P}\omega$-coding of subsets of ω. To avoid confusion, our coding space will be denoted by $\mathcal{O}\omega$ instead of $\mathcal{P}\omega$ (the intended meaning behind the notation $\mathcal{O}\omega$ is that our coding space $\mathcal{P}\omega$ is not only the power set of ω itself, but also topologized as the hyperspace of open subsets of ω).

Definition 2.1. A set X is *$\mathcal{O}\omega$-represented* (or simply, *represented*) if there is a (possibly, multi-valued) partial surjection $\rho :\subseteq \mathcal{O}\omega \to X$. For an $x \in X$, any element of $\rho^{-1}\{x\}$ is called a *ρ-semicode* or a *ρ-name* of x. For a class $\mathbf{E}$, we say that $x \in X$ is *$\mathbf{E}$-semicoded (w.r.t. ρ)* if x has a ρ-semicode $p_x \in \mathbf{E} \cap \mathcal{O}\omega$. For any represented set $\mathcal{X} = (X, \rho)$, we write $\mathcal{X}^{\mathbf{E}}$ for the collection of all $\mathbf{E}$-semicoded elements of $\mathcal{X}$.

The set $\mathbf{E}$ represents our ground model. If the reader is only interested in computability theory and algorithmic randomness theory, we may fix our model $\mathbf{E}$ as the set $\mathbf{E}_{\rm ce}$ consisting of all c.e. subsets of ω. For other instances, we may use $\mathbf{E}_{\Pi^1_1} := \Pi^1_1 \cap \mathcal{O}\omega$, the set of all Π^1_1 subsets of ω, as the universe of hyperarithmetic theory, $\mathbf{E}_{\rm ITTM} := \Sigma_1(L_\lambda) \cap \mathcal{O}\omega$, the set of all infinite-time-Turing-machine (ITTM) semicomputable subsets of ω, as the universe of ITTM computability theory (see [77, 76]).

Example 2.2 (Cantor Space)**.** *Cantor space* 2^ω is the set of infinite binary sequences equipped with the canonical product topology. A basic open set on 2^ω is a cylinder $[\sigma] = \{x \in 2^\omega : \sigma \prec x\}$ for a finite binary string $\sigma \in 2^{<\omega}$. Then 2^ω is represented by $\rho_{2^\omega} :\subseteq \mathcal{O}\omega \to 2^\omega$ such that

$$\rho_{2^\omega}(p) = x \iff p = \{\ulcorner \sigma \urcorner \in \omega : x \in [\sigma]\},$$

where $\ulcorner \cdot \urcorner : 2^{<\omega} \to \omega$ is a fixed computable bijection. Baire space ω^ω also has a similar representation ρ_{ω^ω}.

Under the representation ρ_{2^ω} (ρ_{ω^ω}, resp.), for a point $x \in 2^\omega$ ($x \in \omega^\omega$, resp.), $\mathbf{E}_{\rm ce}$-semicodability is equivalent to computability, $\mathbf{E}_{\Pi^1_1}$-semicodability is equivalent to hyperarithmetical definability, and $\mathbf{E}_{\rm ITTM}$-semicodability is equivalent to ITTM computability.

Example 2.3 (Borel Codes)**.** For a fixed countable basis $\{B_n\}_{n\in\omega}$ of an underlying space $\mathcal{X}$ (e.g., Baire space ω^ω) one can consider a $\mathcal{O}\omega$-representation of the class of all Borel sets (or all $\mathbf{\Sigma}^0_\alpha$ sets for a fixed rank $\alpha < \omega_1$) in $\mathcal{X}$. Formally, we introduce codings (representations) $\sigma^0_n : \mathcal{O}\omega \to \mathbf{\Sigma}^0_n(\mathcal{X})$ and $\pi^0_n : \mathcal{O}\omega \to \mathbf{\Pi}^0_n(\mathcal{X})$ in the following inductive way:

$$\sigma^0_1(p) = \bigcup_{n\in p} B_n, \quad \pi^0_n(p) = \mathcal{X} \setminus \sigma^0_n(p), \quad \sigma^0_n(p) = \bigcup_k \pi^0_n(p^{[k]}).$$

Here, the k-th section $p^{[k]}$ is defined by $\{m : \langle k, m\rangle \in p\}$. One can easily extend this representation to ordinal ranks by using a naming system of ordinals such as a tree-representation system of countable ordinals or Kleene's system $\mathcal{O}$ of ordinal notations. Moreover, (standard) Borel codings are defined by $\sigma^{\rm tt}_\alpha = \sigma^0_\alpha \circ {\rm rng} \circ \rho_{\omega^\omega}$ and $\pi^{\rm tt}_\alpha = \pi^0_\alpha \circ {\rm rng} \circ \rho_{\omega^\omega}$, where ${\rm rng}(q) = \{q(n) : n \in \omega\}$ for $q \in \omega^\omega$.

If $A \subseteq \mathcal{X}$ is $\mathbf{\Sigma}^0_\alpha$ ($\mathbf{\Pi}^0_\alpha$, resp.), we call each element of $(\sigma^0_\alpha)^{-1}\{A\}$ ($(\pi^0_\alpha)^{-1}\{A\}$, resp.) a *Borel semicode* of A, and each element of $(\sigma^{\rm tt}_\alpha)^{-1}\{A\}$ ($(\pi^{\rm tt}_\alpha)^{-1}\{A\}$, resp.) a *Borel code* of A. We sometimes say $\mathbf{\Sigma}^0_\alpha$-code, $\mathbf{\Sigma}^0_\alpha$-semicode, etc. for such codes.

Example 2.4. A set $S \subseteq \mathcal{X}$ is usually called *c.e. open* or *lightface* Σ^0_1 if $S \in (\mathbf{\Sigma}^0_1(\mathcal{X}), \sigma^0_1)^{\mathbf{E}_{\rm ce}}$; that is, $S = \sigma^0_1(p)$ for some c.e. set $p \subseteq \omega$. Every c.e. open set S always has a c.e. Borel code, that is, $S = \sigma^{\rm tt}_1(p)$ for some c.e. set $p \in \mathcal{O}\omega$. A set $S \subseteq \mathcal{X}$ is usually called *co-c.e. closed* or *lightface* Π^0_1 if $S \in (\mathbf{\Pi}^0_1(\mathcal{X}), \pi^0_1)^{\mathbf{E}_{\rm ce}}$; that is, $S = \pi^0_1(p)$ for some c.e. set $p \subseteq \omega$. In other words, it is the complement of a c.e. open set.

Another important effectivization of an open set and a closed set are a Π^1_1-open set and a Σ^1_1-closed set, respectively (see also [57, Section 9]). Here, a set $S \subseteq \mathcal{X}$ is Π^1_1*-open* if $S \in (\mathbf{\Sigma}^0_1(\mathcal{X}), \sigma^0_1)^{\mathbf{E}_{\Pi^1_1}}$; that is, $S = \sigma^0_1(p)$ for some Π^1_1 set $p \subseteq \omega$, and a set $S \subseteq \mathcal{X}$ is Σ^1_1*-closed* if $S \in (\mathbf{\Pi}^0_1(\mathcal{X}), \pi^0_1)^{\mathbf{E}_{\Pi^1_1}}$; that is, $S = \pi^0_1(p)$ for some Π^1_1 set $p \subseteq \omega$. In other words, it is the complement of a Π^1_1-open set. In general, a Π^1_1-open set may not have a Π^1_1-code w.r.t. Borel coding, that is, there is a Π^1_1 set $p \in \mathcal{O}\omega$ such that $\sigma^0_1(p) \neq \sigma^{\rm tt}_1(q)$ for any Π^1_1 set $q \in \mathcal{O}\omega$.

Example 2.5. We often use the symbol $\mathcal{O}\mathcal{X}$ to denote the represented space $(\mathbf{\Sigma}^0_1(\mathcal{X}), \sigma^0_1)$, the hyperspace of open subsets of $\mathcal{X}$. For instance, the topology on the space $\mathcal{O}\omega$ is generated by open sets of the form

$B_e^{\mathcal{O}\omega} = \{x \in \mathcal{O}\omega : D_e \subseteq x\} \in \mathcal{O}\mathcal{O}\omega$, where $(D_e)_{e\in\omega}$ is a computable enumeration of all finite subsets of ω, and then $\mathcal{O}\mathcal{O}\omega$ is represented as follows:

$$\rho_{\mathcal{O}\mathcal{O}\omega}(p) = U \iff U = \bigcup_{e\in p}\{x \in \mathcal{O}\omega : D_e \subseteq x\}.$$

The notion of computability and relativization in represented spaces can be thought of as a special case of the usual *model-theoretic relativization*. Clearly, however, we do not require the whole structure of $\mathbf{E}$ to define the notion of $\mathbf{E}$-relativization $\mathcal{X}^{\mathbf{E}}$, since the definition of $\mathcal{X}^{\mathbf{E}}$ refers only $\mathcal{O}\omega$ (i.e., semicodes) contained in $\mathbf{E}$. Thus, the $\mathcal{O}\omega$-encodability is useful for avoiding any set-theoretic and model-theoretic arguments.

Remark 2.6. The general theory of $\mathcal{O}\omega$-encodability (representability) has been extensively studied in computable analysis under the name of *represented space* (see [75, 11]). In computable analysis, we often use ω^ω-representations rather than $\mathcal{O}\omega$-representations. It makes no difference when we work only on ω-normed pointclasses; e.g., a c.e. $\mathcal{O}\omega$-representation (e.g., a c.e. set generating an open set) is usually identified with a computable ω^ω-representation (e.g., a computable sequence generating an open set), by interchanging a c.e. set $W = \{f(s)\}_{s\in\omega} \in \mathcal{O}\omega$ and its enumeration procedure $\lambda s.f(s) \in \omega^\omega$. However, for a Spector pointclass Γ such as coanalytic sets ($\mathbf{\Pi}^1_1$), a Γ-semicode in $\mathcal{O}\omega$-representation (e.g., a Π^1_1 set generating an open set) may require a Δ-code in ω^κ-representation (e.g., an $\omega_1^{\rm CK}$-computable sequence generating an open set) for $\kappa > \omega$.

Indeed, as a code of a topological space, an $\mathcal{O}\omega$-representation is more straightforward than an ω^ω-representation. For instance, it is known that every T_0 space X having a countable cs-network $\mathcal{N} = (N_i)_{i\in\omega}$ (see [49, 30]) has an (admissible) representation $\rho_X :\subseteq \mathcal{O}\omega \to X$ where $\rho(p) = x$ iff $x \in \bigcap_{i\in p} N_i$ and $\{N_i : i \in p\}$ is a cs-network of X at x (see [69]). The class of admissibly represented spaces (with sequentially continuous maps) forms a Cartesian closed category, and is much larger than the class of second-countable T_0 spaces.

2.1.2. *Coding of Function Spaces.* Here we review definitions of representations of functions spaces (see also [69, 75]).

Definition 2.7. We say that a function $f : \mathcal{O}\omega \to \mathcal{O}\omega$ is *computable* if f is an enumeration operator. Recall that an enumeration operator f is identified with a computable set $\Psi_f \in \mathcal{O}\omega$ of axioms, that is, $n \in f(A)$ iff there is $\langle e, n\rangle \in \Psi_f$ such that $D_e \subseteq A$, where D_e is the e-th finite subset of ω. It is *continuous* if it is computable relative to an oracle, or equivalently, it has an axiom. Then, the space $\mathcal{C}(\mathcal{O}\omega, \mathcal{O}\omega)$ of continuous functions is represented by $\Psi_f \mapsto f$.

Definition 2.8. Let $\mathcal{X} = (X, \rho_X)$ and $\mathcal{Y} = (Y, \rho_Y)$ be represented sets. Then, a partial function $f :\subseteq \mathcal{X} \to \mathcal{Y}$ is *continuously realized* (*computably realized*, resp.) if there exists a continuous (computable, resp.) function $\tilde{f} : \mathcal{O}\omega \to \mathcal{O}\omega$ such that for any ρ_X-name p_x of $x \in \mathrm{dom}(f)$, $\tilde{f}(p_x)$ outputs a ρ_Y-name of $f(x)$. Each axiom Ψ_f of a realizer $\tilde{f}$ of f is also considered as a name of f. Thus, the space $\mathcal{C}(_\subseteq\mathcal{X}, \mathcal{Y})$ of all partial continuously realized function from $\mathcal{X}$ into $\mathcal{Y}$ is represented by the (multi-valued) partial map $\Psi_f \mapsto f$.

For a given function space $\mathcal{F}(_\subseteq\mathcal{X}, \mathcal{Y})$, we sometimes write $\mathcal{F}_{\rm tt}(\mathcal{X}, \mathcal{Y})$ to emphasize that it denotes the space of all $f \in \mathcal{F}(_\subseteq\mathcal{X}, \mathcal{Y})$ with $\mathrm{dom}(f) = \mathcal{X}$. Another important example of a subspace of a function space $\mathcal{C}(_\subseteq\mathcal{X}, \mathcal{Y})$ is the space of uniformly continuous functions defined as follows:

Example 2.9. Let $\mathcal{X} = (X, d_X)$ and $\mathcal{Y} = (Y, d_Y)$ be metric spaces represented by ρ_X and ρ_Y, respectively (see Weihrauch [75]). We say that $\delta \in \omega^\omega$ is a modulus of continuity of a function $f : \mathcal{X} \to \mathcal{Y}$ if $d_X(x, y) < 2^{-\delta(n)}$ implies $d_Y(f(x), f(y)) < 2^{-n}$ for every $x, y \in X$ and $n \in \omega$. The represented space $\mathcal{C}_{\rm uf}(\mathcal{X}, \mathcal{Y})$ of *uniformly continuous functions* from $\mathcal{X}$ into $\mathcal{Y}$ as follows.

$$\rho_{\rm uf}(\langle p, q\rangle) = \rho_C(p), \text{ where } \rho_C \text{ is the induced representation of } \mathcal{C}(\mathcal{X}, \mathcal{Y}),$$
$$\mathrm{dom}(\rho_{\rm uf}) = \{\langle p, q\rangle : \rho_{\omega^\omega}(q) \text{ is a modulus of continuity of } \rho_C(p)\}.$$

Lemma 2.10. *The inclusion map $\mathcal{C}_{\rm tt}(2^\omega, \omega^\omega) \hookrightarrow \mathcal{C}_{\rm uf}(2^\omega, \omega^\omega)$ is computable, that is, there is a computable function mapping each ρ_C-semicode of a function to a $\rho_{\rm uf}$-semicode of the same function.*

Proof. It is well known that there is a computable version of the statement that every continuous function with a compact domain is uniformly continuous. □

Definition 2.11. Suppose that $\mathcal{F}$ is a represented function space, and that x and y are reals. Then, we say that y is $\mathcal{F}$-reducible to x over $\mathbf{E}$ (*written as* $y \leq^{\mathbf{E}}_{\mathcal{F}} x$) if there is a partial function $f \in \mathcal{F}^{\mathbf{E}}$ such that $x \in \mathrm{dom}(f)$ and $f(x) = y$.

Remark 2.12. We use notations $\leq_{\mathrm{T}(\mathbf{E})}$, $\leq_{\mathrm{wtt}(\mathbf{E})}$, and $\leq_{\mathrm{tt}(\mathbf{E})}$ instead of $\leq^{\mathbf{E}}_{\mathcal{C}}$,$\leq^{\mathbf{E}}_{\mathcal{C}_{\mathrm{uf}}}$, and $\leq^{\mathbf{E}}_{\mathcal{C}_{\mathrm{tt}}}$. For instance, if $\mathbf{E}_{\mathrm{ce}} := \Sigma^0_1 \cap \mathcal{O}\omega$ then the reducibility notions $\leq_{\mathrm{T}(\mathbf{E}_{\mathrm{ce}})}$, $\leq_{\mathrm{wtt}(\mathbf{E}_{\mathrm{ce}})}$ and $\leq_{\mathrm{tt}(\mathbf{E}_{\mathrm{ce}})}$ are equivalent to Turing reducibility, weak-truth-table reducibility and truth-table reducibility, respectively.

One can also introduce a represented space of Lipscitz continuous functions to study strong weak truth-table reducibility (also known as computable Lipschitz reducibility). See Downey-Hirschfeldt-LaForte [22].

2.1.3. *Pointclasses and Measurability.* In our paper, a represented function space $\mathcal{F}$ will play a role as a level of uniformity. Usually, our level of uniformity $\mathcal{F}$ is chosen as a subclass of partial functions (that are universally measurable, in most cases) which are not necessarily continuous; e.g., measurable w.r.t. Borel sets, C-sets and R-sets (see [14, 15, 16]). The relationship between several generalized computability notions and classical measure theoretic notions such as C-sets and R-sets has been analyzed by many researchers (see for instance, [1, 9, 33, 34, 47]).

Generally, we have a well-developed theory on $\mathbf{\Gamma}$-measurability for a pointclass $\mathbf{\Gamma}$ (see Moschovakis [55]). Here, a *pointclass* is an operation Γ such that $\Gamma\mathcal{X} \subset \mathcal{P}\mathcal{X}$ for any computable metric space $\mathcal{X}$ ([29, 55]). For instance, Δ^i_α, Σ^i_α and Π^i_α are pointclasses. We often identify a pointclass Γ with the union of all $\Gamma\mathcal{X}$ where $\mathcal{X}$ ranges over all computable metric spaces. Recall that a function $f : \mathcal{X} \to \mathcal{Y}$ is $\mathbf{\Gamma}$-*measurable* if $f^{-1}[U] \in \mathbf{\Gamma}\mathcal{X}$ for each open set $U \in \mathcal{O}\mathcal{Y}$. To define this notion, we only need a pointclass Γ defined on $\mathcal{X}$, and openness on $\mathcal{Y}$. The purpose of this paper is not to establish a new framework of pointclasses on general represented spaces; therefore, we still keep the restriction that pointclasses $\mathbf{\Gamma}$ are defined only on computable metric spaces.

We will see that by using an ω-parametrization of a (lightface) pointclass Γ, the boldface pointclass $\mathbf{\Gamma}$ and the associated function space $\mathcal{F}_{\mathbf{\Gamma}}$ can be viewed as $\mathcal{O}\omega$-represented spaces.

Definition 2.13 (see Moschovakis [55])**.** Let $\mathcal{Z}$ be a set. A pointclass Γ is $\mathcal{Z}$*-parametrized* if for any separable metric space $\mathcal{X}$, there is a surjection $\rho_{\Gamma\mathcal{X}} : \mathcal{Z} \to \Gamma\mathcal{X}$ such that

$$\{(x, y) \in \mathcal{Z} \times \mathcal{X} : y \in \rho_{\Gamma\mathcal{X}}(x)\} \in \Gamma.$$

If a pointclass Γ is ω-parametrized, the corresponding *boldface* pointclass $\mathbf{\Gamma}$ is defined as follows:

$$A \in \mathbf{\Gamma}\mathcal{X} \iff (\exists B \in \Gamma(\omega^\omega \times \mathcal{X}))(\exists p \in \omega^\omega)\ A = \{x \in \mathcal{X} : (p, x) \in B\}.$$

Then, $\mathbf{\Gamma}\mathcal{X}$ is ω^ω-represented by $\tilde{\rho}_{\Gamma\mathcal{X}} : \langle e, p\rangle \mapsto A$, where $e \in \omega$ is a $\rho_{\Gamma(\omega^\omega \times \mathcal{X})}$-name of B. Then, an $\mathcal{O}\omega$-representation of $\mathbf{\Gamma}\mathcal{X}$ is given by $\hat{\rho}_{\Gamma\mathcal{X}} = \tilde{\rho}_{\Gamma\mathcal{X}} \circ \rho_{\omega^\omega}$.

In this paper, the codomain $\mathcal{Y}$ of a $\mathbf{\Gamma}$-measurable function is allowed to be a non-admissibly represented space such as the space $\mathbf{\Pi}^0_2(2^\omega)$ of G_δ subsets of Cantor space (represented by Borel coding). However, if $\mathcal{Y}$ is non-admissible, the space $\mathcal{O}\mathcal{Y}$ is ill-behaved. Instead of using $\mathcal{O}\mathcal{Y}$, we consider a $\mathbf{\Gamma}$-measurable realizer. Recall that if underlying spaces are second-countable and $\mathbf{\Gamma}$ is well-behaved, then $f : \mathcal{X} \to \mathcal{Y}$ is $\mathbf{\Gamma}$-measurable if and only if $f^{-1} : \mathcal{O}\mathcal{Y} \to \mathbf{\Gamma}\mathcal{X}$ sending $U \mapsto f^{-1}[U]$ has a continuous realizer (see [10, 42]).

Definition 2.14. Let $\mathbf{\Gamma}$ be a ω-parametrized pointclass, and $\mathcal{X}$ be a separable metric space. Then, a function $f : \mathcal{X} \to \mathcal{O}\omega$ is $\mathbf{\Gamma}$*-measurable* if the function $f^{-1} : \mathcal{O}\mathcal{O}\omega \to \mathbf{\Gamma}\mathcal{X}$ sending $U \mapsto f^{-1}[U]$ has a continuous realizer. If $\mathbf{\Gamma}$ is sufficiently well-behaved (e.g., Γ is a Kleene-or-Spector pointclass), $f : \mathcal{X} \to \mathcal{O}\omega$ is $\mathbf{\Gamma}$-measurable if and only if the following set G_f belongs to the pointclass $\mathbf{\Gamma}(\mathcal{X} \times \omega)$:

$$G_f := \{(x, n) \in \mathcal{X} \times \omega : n \in f(x)\}.$$

Thus, the space $\mathcal{F}_{\mathbf{\Gamma}}(\mathcal{X}, \mathcal{O}\omega)$ of all $\mathbf{\Gamma}$-measurable functions from $\mathcal{X}$ into $\mathcal{O}\omega$ is represented by a map sending each $\mathbf{\Gamma}$-code of G_f to f. Generally, for a represented space $\mathcal{Y} = (Y, \rho_Y)$, a function $f : \mathcal{X} \to \mathcal{Y}$ is $\mathbf{\Gamma}$*-measurable* if f has a $\mathbf{\Gamma}$-measurable realizer in the sense that $f = \rho_Y \circ \hat{f}$ for some $\mathbf{\Gamma}$-measurable function $\hat{f} : \mathcal{X} \to \mathcal{O}\omega$. The space $\mathcal{F}_{\mathbf{\Gamma}}(\mathcal{X}, \mathcal{Y})$ of all $\mathbf{\Gamma}$-measurable functions from $\mathcal{X}$ into $\mathcal{Y}$ is represented by a map sending each $\mathbf{\Gamma}$-code of $G_{\hat{f}}$ to f.

Given a pointclass Γ, a set $p \in \mathcal{O}\omega$ belongs to Γ if $p \in \Gamma\omega$. We use all Γ-names $\mathbf{E}_\Gamma := \Gamma\omega \subseteq \mathcal{O}\omega$ as our ground model, and $\mathcal{F}_{\mathbf{\Gamma}}$ as an associated level of uniformity. If we work on computability theory (hyperarithmetic theory, resp.), we may choose the pointclass Σ^0_1 (Π^1_1, resp.), which generates the ground model $\mathbf{E}_{\rm ce} := \Sigma^0_1 \cap \mathcal{O}\omega$ ($\mathbf{E}_{\Pi^1_1} := \Pi^1_1 \cap \mathcal{O}\omega$, resp.) and the space $\mathcal{C}$ of partial continuous functions (the space $\mathcal{F}_{\mathbf{\Pi}^1_1}$ of partial $\mathbf{\Pi}^1_1$-measurable functions, resp.) In either case, our ground model $\mathbf{E}$ is exactly of the form $\Sigma_1(\mathbf{M}) \cap \mathcal{O}\omega$, the collection of all Σ_1-definable subsets of ω over a model $\mathbf{M}$ of Kripke-Platek set theory. However, a model $\mathbf{M} \models \mathsf{KP}$ itself does not involve a function space. Our ground model $\mathbf{E}$ should involve a (represented) function space $\mathcal{F}$ to define the corresponding reducibility notion. This is the reason why we need a pointclass rather than a model of KP.

Definition 2.15. If Γ is an ω-parametrized pointclass, we write $x \leq_\Gamma y$ for $x \leq^{\mathbf{E}_\Gamma}_{\mathcal{F}_{\mathbf{\Gamma}}} y$ (see Definition 2.11).

Example 2.16. Reducibilities $\leq_{\Sigma^0_1}$ and $\leq_{\Pi^1_1}$ (restricted to Cantor space, Baire space or Euclidean n-space for $n \in \omega$) are Turing reducibility $\leq_{\rm T}$ and hyperarithmetical reducibility $\leq_{\rm h}$ while $\leq_{{\rm T}(\mathbf{E}_{\Pi^1_1})} := \leq^{\mathbf{E}_{\Pi^1_1}}_{\mathcal{C}}$ is higher-Turing reducibility ([5]).

2.1.4. *Kleene Pointclasses and Spector Pointclasses.* A well-behaved level of uniformity $\mathcal{F}_{\mathbf{\Gamma}}$ is usually defined as $\mathbf{\Gamma}$-measurability for a *Spector pointclass* Γ (see [54, 55, 41, 40]); e.g., $\Gamma = \Pi^1_1$ (lightface coanalytic sets), $\Gamma = {}_2{\rm env}({\rm E}_1)$ (the 2-envelop, all semi-computable pointsets, of Tugué's type 2 functional ${\rm E}_1$), Γ = all pointsets semi-computable by infinite time Turing machines (see [76]), etc. By Moschovakis' companion theorem ([54, Theorem 9E.1]; see also [31]), every Spector pointclass Γ involves a transitive model $\mathbf{M}_\Gamma$ of Kripke-Platek set theory KP such that the $\mathbf{M}_\Gamma$ Σ_1-projects to ω and $\Sigma_1(\mathbf{M}_\Gamma)$-definable set $A \in \mathcal{O}\omega$ are exactly Γ subsets of ω. For instance, the companion of the Spector pointclass Π^1_1 is $L_{\omega_1^{\rm CK}}$.

We say that Γ is a *Kleene-or-Spector pointclass* if it is a relativized Kleene Σ-pointclass (i.e., $\Sigma^0_\alpha(z)$ for some $\alpha < \omega_1^{{\rm CK},z}$) or a Spector pointclass. In particular, Γ admits an ω-parametrization, Γ-norm, and uniformization (see [55]). The notion of a Kleene-or-Spector pointclass is useful for developing algorithmic randomness theory, hyperarithmetical randomness theory, infinite-time-Turing-machine (ITTM) randomness theory, etc. in a unified manner. We say that a set $\mathbf{E} \subseteq \mathcal{O}\omega$ is *principal KS* (or a topped KS-model) if $\mathbf{E} = \mathbf{E}_\Gamma$ for a Kleene-or-Spector pointclass Γ. However, generalized recursion theory is not restricted to principal KS models; e.g., arithmetical definability, infinite-time-register-machine (ITRM) computability, and constructibility (in the sense of Gödel's L). We introduce the following notion to capture such generalized computability notions:

Definition 2.17. A *KS base* of Γ is a collection $(\Gamma_u)_{u\in\Lambda}$ of Kleene-or-Spector pointclasses such that

(i) $\Gamma_u\mathcal{X} \subseteq \Gamma\mathcal{X}$ for all $u \in \Lambda$.
(ii) (Localization) for all $A \in \Gamma\mathcal{X}$ there is $u \in \Lambda$ such that $A \in \Gamma_u\mathcal{X}$.
(iii) (Amalgamation) for all $u, v \in \Lambda$ there is $w \in \Lambda$ such that $\Gamma_u \cup \Gamma_v \subseteq \Gamma_w$.

A pointclass Γ is *locally KS* if it has a KS base. A set $\mathbf{E} \subseteq \mathcal{O}\omega$ is *locally KS* if $\mathbf{E} = \mathbf{E}_\Gamma$ for a locally KS pointclass. If Γ has an $\mathcal{O}\omega$-indexed KS base $(\Gamma_u)_{u\in\mathcal{O}\omega}$, then we may define the represented space $\mathcal{F}_{\mathbf{\Gamma}}$ (whose representation depends on the choice of such a base) as follows: $\langle u, q\rangle$ names f iff q is a name of f in the represented space $\mathcal{F}_{\Gamma_u}$.

Example 2.18. The pointclass $\Delta^0_{<\omega}$ has a KS base $(\Sigma^0_n)_{n<\omega}$, which generates the universe of arithmetical sets and arithmetical reducibility. The pointclass Δ^1_1 has a KS base $(\Sigma^0_\alpha)_{\alpha<\omega_1^{\rm CK}}$. The pointclass corresponding to ITRM semi-computability has a KS base $(\partial(\Sigma^0_1)_n)_{n<\omega}$, where $(\Sigma^0_1)_n$ is the n-th level of the difference hierarchy of Δ^0_2.

It is well known that basic computability theoretic results hold for Kleene-or-Spector pointclasses (see [25, 55]). Based on the same idea, one can develop computability theory for a locally KS-set in a straightforward manner. For instance, it is very easy to show characterizations of $\mathbf{E}$-Martin-Löf/Schnorr randomness via complexity, martingale, etc. for any locally KS-set $\mathbf{E}$.

2.2. Ideals and Randomness.

2.2.1. *Coding of Ideals.* A representation (a coding) of an ideal (usually associated with a forcing notion, in the context of idealized forcing [78]) is often used to define the notion of randomness (and genericity) over a model $\mathbf{M} \models \mathsf{ZFC}$. We are mostly interested in three Borel σ-ideals $\mathcal{N}$, $\mathcal{M}$, and $\mathcal{E}$ generated by Lebesgue null sets, meager sets, and closed null sets. In set theory, these ideals are naively represented by Borel codes, or more accurately, G_δ-codes (for $\mathcal{N}$) and F_σ-codes (for $\mathcal{M}$ and $\mathcal{E}$). However, in algorithmic randomness theory, we require more delicate handling of representations of the ideal $\mathcal{N}$ of Lebesgue null sets (see [44] and Definition 2.22). Thus, each ideal may have various different representations.

As mentioned before, it is natural to choose multi-valued maps to represent ideals. Given a representation $\gamma :\subseteq \mathcal{O}\omega \to X$, we define the multi-representation ${}^{\subseteq}\gamma :\subseteq \mathcal{O}\omega \to X$ by declaring that ${}^{\subseteq}\gamma(x)$ names y iff $y \subseteq \gamma(x)$. The σ-ideal $\mathcal{N}$ can be represented by (restrictions of) ${}^{\subseteq}\pi^0_2$ or ${}^{\subseteq}\pi^{\rm tt}_2$, and $\mathcal{M}$ and $\mathcal{E}$ by ${}^{\subseteq}\sigma^0_2$ and ${}^{\subseteq}\sigma^{\rm tt}_2$ in the following manner:

Definition 2.19 (Induced Representation of Ideals)**.** Suppose that a pointclass $\mathbf{\Gamma}\omega^\omega$ is represented by $\gamma :\subseteq \mathcal{O}\omega \to \mathbf{\Gamma}\omega^\omega$, and a σ-ideal $\mathcal{J} \subsetneq \mathcal{P}(\omega^\omega)$ is endowed with its generator $\hat{\mathcal{J}} \subset \mathbf{\Gamma}$. Then, one can automatically obtain the *induced multi-representation* $\gamma \downharpoonleft \hat{\mathcal{J}} :\subseteq \omega^\omega \to \mathcal{J}$ by restricting γ as follows:

$$(\gamma \downharpoonleft \hat{\mathcal{J}})(p) = {}^{\subseteq}\bigcup_{k\in\omega} \gamma(p^{[k]}), \quad \mathrm{dom}(\gamma \downharpoonleft \hat{\mathcal{J}}) = \{p : \gamma(p^{[k]}) \in \hat{\mathcal{J}} \text{ for all } k \in \omega\}.$$

Here, ${}^{\subseteq}\bigcup_k \gamma(p^{[k]})$ is an abbreviation of ${}^{\subseteq}\delta$, where $\delta : p \mapsto \bigcup_k \gamma(p^{[k]})$.

Example 2.20. The σ-ideals $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{E}$ are σ-generated by Lebesgue null G_δ sets $\hat{\mathcal{N}}$, nowhere dense closed sets $\hat{\mathcal{M}}$ and Lebesgue null closed sets $\hat{\mathcal{E}}$, respectively.

Example 2.21. Recall that $\mathcal{X}^{\mathbf{E}}$ is the set of all $\mathbf{E}$-semicoded elements of a represented set $\mathcal{X}$. If $\mathcal{X}$ is multi-represented, then a name p may code many elements of $\mathcal{X}$. For instance, if p is a c.e. semi-code of a G_δ set $\pi^0_2(p)$, then p codes all subsets of the Π^0_2 sets $\pi^0_2(p)$ w.r.t. the multi-representation ${}^{\subseteq}\pi^0_2$. Therefore, $(\mathcal{N}, \pi^0_2 \downharpoonleft \mathcal{N})^{\mathbf{E}_{\rm ce}}$ is exactly the set of all sets A such that $A \subseteq B$ for some lightface Π^0_2 null set B. Similarly, $(\mathcal{E}, \pi^0_1 \downharpoonleft \mathcal{M})^{\mathbf{E}_{\Pi^1_1}}$ is exactly the set of all sets A such that $A \subseteq B$ for some Σ^1_1-closed null set B.

Definition 2.22 (see also [44])**.** Let λ be the Lebesgue measure on 2^ω. We define two representations $\rho_{\rm MLR}$ and $\rho_{\rm SR}$ of the ideal $\mathcal{N}$, and a representation $\rho_{\rm WR}$ of the ideal $\mathcal{E}$, where recall that $\mathcal{E}$ is the σ-ideal generated by all closed null sets. The *Martin-Löf representation* (or ML-representation) $\rho_{\rm MLR} :\subseteq \mathcal{O}\omega \to \mathcal{N}$, the *Schnorr representation* $\rho_{\rm SR} :\subseteq \mathcal{O}\omega \to \mathcal{N}$, and the *Kurtz representation* $\rho_{\rm WR} :\subseteq \mathcal{O}\omega \to \mathcal{E}$ are defined as follows:

$$\rho_{\rm MLR}(p) = {}^{\subseteq}\pi^0_2(p) = {}^{\subseteq}\bigcap_n \sigma^0_1(p^{[n]}), \qquad \mathrm{dom}(\rho_{\rm MLR}) = \{p : \lambda(\sigma^0_1(p^{[n]})) \leq 2^{-n} \text{ for all } n \in \omega\}.$$

$$\rho_{\rm SR}(p) = {}^{\subseteq}\pi^0_2(p) = {}^{\subseteq}\bigcap_n \sigma^0_1(p^{[n]}), \qquad \mathrm{dom}(\rho_{\rm SR}) = \{p : \lambda(\sigma^0_1(p^{[n]})) = 2^{-n} \text{ for all } n \in \omega\}.$$

$$\rho_{\rm WR}(p) = {}^{\subseteq}\sigma^0_2(p) = {}^{\subseteq}\bigcup_n \pi^0_1(p), \qquad \mathrm{dom}(\rho_{\rm WR}) = \{p : \lambda(\pi^0_1(p^{[n]})) = 0 \text{ for all } n \in \omega\}.$$

We use $\mathcal{N}_{\rm MLR}$, $\mathcal{N}_{\rm SR}$ and $\mathcal{E}_{\rm WR}$ to denote $(\mathcal{N}, \rho_{\rm MLR})$, $(\mathcal{N}, \rho_{\rm SR})$ and $(\mathcal{E}, \rho_{\rm WR})$, respectively.

Example 2.23. An element in $(\mathcal{N}_{\rm MLR})^{\mathbf{E}_{\rm ce}}$, $(\mathcal{N}_{\rm SR})^{\mathbf{E}_{\rm ce}}$, and $(\mathcal{N}_{\rm WR})^{\mathbf{E}_{\rm ce}}$ can be identified with a Martin-Löf test, a Schnorr test, and a Kurtz test, respectively. An element in $(\mathcal{N}_{\rm MLR})^{\mathbf{E}_{\Pi^1_1}}$ is also known as a Π^1_1-Martin-Löf test. Note that $\rho_{\rm SR}$ is essentially equivalent to the representation of $\mathcal{N}$ introduced by Pawlikowski-Recław [60]. The Kurtz representation $\rho_{\rm WR}$ is equal to the induced representation $\pi^0_1 \downharpoonleft \hat{\mathcal{E}}$ (in the sense of Example 2.19) of $\mathcal{E}$.

Definition 2.24. The *weak-1-genericity representation* $\rho_{\rm W1G} :\subseteq \mathcal{O}\omega \to \mathcal{M}$ and the *Cohen forcing representation* $\rho_{\rm 1G} :\subseteq \mathcal{O}\omega \to \mathcal{M}$ are defined as follows:

$$\rho_{\rm W1G}(p) = {}^{\subseteq}\sigma^0_2(p) = {}^{\subseteq}\bigcup_n \pi^0_1(p^{[n]}), \quad \mathrm{dom}(\rho_{\rm W1G}) = \{p : \pi^0_1(p^{[n]}) \text{ is nowhere dense for all } n \in \omega\}.$$

$$\rho_{\rm 1G}(p) = {}^{\subseteq}\bigcup_n \partial(\sigma^0_1(p^{[n]})), \text{ where } \partial U \text{ is the boundary of } U.$$

Remark 2.25. The weakly-1-generic representation ρ_{W1G} is equivalent to the induced representation (in the sense of Definition 2.19) of $\mathcal{M}$.

2.2.2. *Genericity and Randomness.* The notion of a represented ideal automatically involves the notion of quasi-genericity.

Definition 2.26. Suppose that $\mathcal{J}$ is a (multi-)represented Borel ideal in an underlying topological space $\mathcal{X}$. A point $z \in \mathcal{X}$ is said to be *$\mathcal{J}$-quasigeneric (or $\mathcal{J}$-random) over* $\mathbf{E}$ if z avoids all $\mathbf{E}$-semicoded $\mathcal{J}$-negligible sets, that is, $\{z\} \notin \mathcal{J}^{\mathbf{E}}$. In this case, we write $z \in \mathsf{RND}^{\mathbf{E}}_{\mathcal{J}}$.

Example 2.27 (Genericity w.r.t. Induced Representations)**.** As computability-theoretic examples, weak 2-randomness is $(\mathcal{N}, \pi^0_2 \downharpoonright \mathcal{N})$-quasigenericity over $\mathbf{E}_{\mathrm{ce}}$, Kurtz randomness is $(\mathcal{E}, \pi^0_1 \downharpoonright \mathcal{E})$-quasigenericity over $\mathbf{E}_{\mathrm{ce}}$, weak 1-genericity is $(\mathcal{M}, \pi^0_1 \downharpoonright \mathcal{M})$-quasigenericity over $\mathbf{E}_{\mathrm{ce}}$, and Σ^1_1-Kurtz randomness is $(\mathcal{E}, \pi^0_1 \downharpoonright \mathcal{E})$-quasigenericity over $\mathbf{E}_{\Pi^1_1}$.

As set-theoretic examples, Cohen (quasi-)genericity over $\mathbf{M} \models \mathsf{ZF}$ in the sense of set theory is $(\mathcal{M}, \pi^{\mathrm{tt}}_1 \downharpoonright \mathcal{M})$-quasigenericity over $\mathcal{O}\omega \cap \mathbf{M}$, and randomness over $\mathbf{M} \models \mathsf{ZF}$ is $(\mathcal{N}, \pi^{\mathrm{tt}}_2 \downharpoonright \mathcal{N})$-quasigenericity over $\mathcal{O}\omega \cap \mathbf{M}$. Here, in set theoretic context, we normally use Borel codes such as π^{tt}_2 rather than Borel semi-codes such as π^0_2 (in the sense of Example 2.3), though there is no difference between them under full separation. However, generally, they define different quasigenericity. For instance, $(\mathcal{N}, \pi^{\mathrm{tt}}_2 \downharpoonright \mathcal{N})$-quasigenericity over $\mathbf{E}_{\Pi^1_1}$ is exactly randomness over $L_{\omega_1^{\mathrm{CK}}}$ (i.e., Δ^1_1-randomness), whereas $(\mathcal{N}, \pi^0_2 \downharpoonright \mathcal{N})$-quasigenericity over $\mathbf{E}_{\Pi^1_1}$ is higher weak 2-randomness [5].

Example 2.28 (Genericity in Computability Theory)**.** Martin-Löf randomness is equivalent to $(\mathcal{N}, \rho_{\mathrm{MLR}})$-quasigenericity over $\mathbf{E}_{\mathrm{ce}}$. Schnorr randomness is equivalent to $(\mathcal{N}, \rho_{\mathrm{SR}})$-quasigenericity over $\mathbf{E}_{\mathrm{ce}}$. Moreover, 1-genericity is equivalent to $(\mathcal{M}, \rho_{\mathrm{1G}})$-genericity over $\mathbf{E}_{\mathrm{ce}}$.

The notion of *relative randomness* is one of the main concepts in algorithmic randomness theory. This notion is generalized as follows:

Definition 2.29. We say that a point z is *$\mathcal{J}$-quasigeneric* (or *$\mathcal{J}$-random*) *$\mathcal{F}$-uniformly relative to y over* $\mathbf{E}$ (denoted by $z \in \mathsf{RND}^{\mathcal{F},\mathbf{E}}_{\mathcal{J}}(y)$) if z avoids all $\mathcal{J}$-negligible sets $J \leq^{\mathbf{E}}_{\mathcal{F}} y$, that is,

$$J \in \mathcal{F}^{\mathbf{E}}(_{\subset}2^{\omega}, \mathcal{J}) \text{ and } y \in \mathrm{dom}(J) \implies z \notin J(y).$$

Each map $J \in \mathcal{F}^{\mathbf{E}}(_{\subset}2^{\omega}, \mathcal{J})$ is called an *$\mathcal{F}$-uniform $\mathcal{J}$-test over* $\mathbf{E}$. For instance, a map in $\mathcal{C}^{\mathbf{E}_{\mathrm{ce}}}(_{\subset}2^{\omega}, \mathcal{N}_{\mathrm{MLR}})$ is usually referred as an *oracle Martin-Löf test*, and a map in $\mathcal{C}^{\mathbf{E}_{\mathrm{ce}}}_{\mathrm{tt}}(_{\subset}2^{\omega}, \mathcal{N}_{\mathrm{SR}})$ is referred as a *truth-table Schnorr test* or a *uniform Schnorr test* (see [28, 43, 44, 51]).

2.3. Lowness for Randomness.

Lowness for tests and randomness is an important notion in algorithmic randomness theory (see [24, 27, 57]). Now we generalize the notion of lowness for (uniform) tests.

Definition 2.30. A real $z \in 2^{\omega}$ is *low for $\mathcal{F}$-uniform $(\mathcal{I}, \mathcal{J})$-tests over* $\mathbf{E}$ (or $(\mathcal{I}, \mathcal{J}; \mathcal{F}, \mathbf{E})$-low$_{\mathrm{test}}$) if every $\mathcal{J}$-negligible set $J \leq^{\mathbf{E}}_{\mathcal{F}} z$ is also $\mathcal{I}$-negligible over $\mathbf{E}$, i.e.,

$$J \in \mathcal{F}^{\mathbf{E}}(_{\subset}2^{\omega}, \mathcal{J}) \implies J(z) \in \mathcal{I}^{\mathbf{E}}.$$

In the theory of algorithmic randomness, there are many kinds of results of the form that lowness for randomness tests is equivalent to lowness for randomness. To introduce the notion of uniform lowness for randomness, we define the uniform genericity notion as a relativization of the usual genericity $\mathsf{RND}^{\mathbf{E}}_{\mathcal{J}}$ in Definition 2.26.

Definition 2.31. We say that a real $z \in 2^{\omega}$ is *low for $(\mathcal{I}, \mathcal{J})$ w.r.t. $\mathcal{F}$-uniform relativization over* $\mathbf{E}$ (or $(\mathcal{I}, \mathcal{J}; \mathcal{F}, \mathbf{E})$-low) if every $\mathcal{I}$-quasigeneric real over $\mathbf{E}$ is $\mathcal{J}$-quasigeneric $\mathcal{F}$-uniformly relative to $A \in 2^{\omega}$ over $\mathbf{E}$, that is, $\mathsf{RND}^{\mathbf{E}}_{\mathcal{I}} \subseteq \mathsf{RND}^{\mathcal{F},\mathbf{E}}_{\mathcal{J}}(A)$ holds.

If Γ is a (Kleene-or-Spector) pointclass, then one may say lowness for $(\mathcal{I}, \mathcal{J})$-randomness over $\mathbf{E}_{\Gamma}$ (lowness for $(\mathcal{I}, \mathcal{J})$-tests over $\mathbf{E}_{\Gamma}$, resp.) instead of $(\mathcal{I}, \mathcal{J}; \mathcal{F}_{\mathbf{\Gamma}}, \mathbf{E}_{\Gamma})$-lowness ($(\mathcal{I}, \mathcal{J}; \mathcal{F}_{\mathbf{\Gamma}}, \mathbf{E}_{\Gamma})$-low$_{\mathrm{test}}$-ness, resp.)

Remark 2.32. Note that a real $z \in 2^\omega$ is low for $(\mathcal{I}, \mathcal{J})$ w.r.t. $\mathcal{F}$-uniform relativization over $\mathbf{E}$ if and only if the following holds:

$$J \in \mathcal{F}^{\mathbf{E}}({}_{\subset}2^\omega, \mathcal{J}) \implies J(z) \subseteq \bigcup \mathcal{I}^{\mathbf{E}}.$$

Therefore, $(\mathcal{I}, \mathcal{J}; \mathcal{F}, \mathbf{E})$-low$_{\text{test}}$-ness implies $(\mathcal{I}, \mathcal{J}; \mathcal{F}, \mathbf{E})$-lowness.

Example 2.33. $(\mathcal{N}_{\text{MLR}}, \mathcal{N}_{\text{MLR}}; \mathcal{C}, \mathbf{E}_{\text{ce}})$-lowness is usually referred as lowness for Martin-Löf randomness, and $(\mathcal{N}_{\text{SR}}, \mathcal{N}_{\text{SR}}; \mathcal{F}_{\mathbf{\Pi}^1_1}, \mathbf{E}_{\Pi^1_1})$-lowness is referred as lowness for Π^1_1-Schnorr randomness.

2.3.1. *Transitive Additivity.* We now present an example of use of *uniform relativization* in set theory. Recall that $\mathsf{add}(\mathcal{I}, \mathcal{J})$ is the smallest cardinal such that the union of an A-indexed collection of $\mathcal{I}$-negligible sets is still $\mathcal{J}$-negligible whenever the cardinality of A is less than $\mathsf{add}(\mathcal{I}, \mathcal{J})$ (see [3]). We may say that A is $\mathcal{F}$-uniformly $(\mathcal{I}, \mathcal{J})$-small if the union of an A-indexed $\mathcal{F}$-uniform collection of $\mathcal{I}$-negligible sets is still $\mathcal{J}$-negligible. More generally, we consider the following notion:

Definition 2.34. Suppose that $\mathcal{F}$ is a (represented) collection of partial functions, and $\mathcal{I}$ and $\mathcal{J}$ are ideals endowed with multi-representations. We say that a set $A \subseteq \omega^\omega$ is *$\mathcal{F}$-uniformly $(\mathcal{I}, \mathcal{J})$-small over* $\mathbf{E}$ (written as $A \in \mathrm{Add}^{\mathcal{F},\mathbf{E}}(\mathcal{I}, \mathcal{J})$) if

$$N \in \mathcal{F}^{\mathbf{E}}({}_{\subset}\omega^\omega, \mathcal{I}) \implies N[A] := \bigcup \{N(x) : x \in A \cap \mathrm{dom}(N)\} \in \mathcal{J}^{\mathbf{E}}.$$

If $\mathcal{I} = \mathcal{J}$, we write $A \in \mathrm{Add}^{\mathcal{F},\mathbf{E}}(\mathcal{I})$. If $\mathcal{F}$ is the collection of all partial functions on $\mathcal{O}\omega$, then we write $A \in \mathrm{Add}^{\mathbf{E}}(\mathcal{I}, \mathcal{J})$. If $\mathbf{E} = \mathbf{V} \cap \mathcal{O}\omega$, we write $A \in \mathrm{Add}^{\mathcal{F}}(\mathcal{I}, \mathcal{J})$.

Remark 2.35. Suppose that $\mathcal{C}$ and $\mathcal{B}$ are collections of all partial continuous and Borel functions, respectively. The notions $\mathrm{Add}^{\mathcal{B}}(\mathcal{J})$ and $\mathrm{Add}^{\mathcal{C}}(\mathcal{J})$ can be equivalent to the notions $\mathrm{Add}^{\dagger}(\mathcal{J})$ and $\mathrm{Add}^{\ddagger}(\mathcal{J})$ introduced by Pawlikowski-Recław [60], respectively, by carefully choosing a representation of $\mathcal{J}$.

Now we introduce the strongest uniformity level of $(\mathcal{I}, \mathcal{J})$-smallness known as *transitive additivity*.

Definition 2.36 (see also [3])**.** Let $\mathcal{I}$ and $\mathcal{J}$ be represented ideals. We say that $X \subseteq 2^\omega$ is *$(\mathcal{I}, \mathcal{J})$-additive over* $\mathbf{E}$ (written as $X \in \mathrm{Add}^{\star,\mathbf{E}}(\mathcal{I}, \mathcal{J})$) if $N \in \mathcal{I}$ implies $X + N \in \mathcal{J}$, where recall that $X + N$ is the set of symmetric differences $A \triangle B$ of all pairs $(A, B) \in X \times N$. We also use the notation $\mathrm{Add}^{\star,\mathbf{E}}(\mathcal{I})$ if $\mathcal{I} = \mathcal{J}$.

2.4. **Traceability and Slalom.** The notion of traceability is a standard and useful tool to characterize lowness for randomness [24, 57]. This notion is also used to characterize transitive additivity notions [3].

Definition 2.37 (see [24, 57])**.** A *trace* (also known as a *slalom* in set theory) is a sequence $T = \langle T_n \rangle_{n\in\omega}$ of finite subsets of ω with $|T_n| \leq 2^n$ for every $n \in \omega$. For $\mathsf{Q} \subseteq \mathcal{P}\omega$, we say that T *Q-often traces* $V \subseteq \omega^\omega$ if

$$(\forall h \in V)\ \{n \in \omega : h(n) \in T_n\} \in \mathsf{Q}.$$

We simply say that T *traces* V (T *i.o. traces* V, resp.) if Q is the set of all cofinite sets Q_{cof} (all infinite sets Q_{inf}, resp.) We also say that for a given $g \in \omega^\omega$, T *g-o. traces* V if $\mathsf{Q} = \mathsf{Q}_g := \{X \subseteq \omega : X \cap [g(k), g(k+1)) \neq \emptyset$, for almost all $k\}$.

One can also introduce $\mathbf{E}$-relativized versions of a *c.e. trace* and a *computable trace* by defining representations of sets $\mathcal{T}$ of slaloms as follows:

Definition 2.38 (Representation of Traces)**.** Let $\mathcal{T}$ be the set of all traces. One can automatically obtain a representation $\tau_{\text{semi}} :\subseteq \mathcal{O}\omega \to \mathcal{T}$ by thinking of $\mathcal{T}$ as a subspace of $\mathcal{O}\omega$, where each $\langle T_n \rangle_{n\in\omega} \in \mathcal{T}$ is identified with $\{\langle n, k \rangle : k \in T_n\} \in \mathcal{O}\omega$. We also consider the following representation $\tau_{\text{tt}} :\subseteq \mathcal{O}\omega \to \mathcal{T}$:

$$\tau_{\text{tt}}(\langle p, q \rangle) = \tau_{\text{semi}}(p) = \langle p^{[n]} \rangle_{n\in\omega}, \quad \mathrm{dom}(\tau_{\text{tt}}) = \{\langle p, q \rangle : \langle p^{[n]} \rangle_{n\in\omega} \in \mathcal{T}, \text{ and } q(n) = |p^{[n]}|\}.$$

Then, the represented spaces $(\mathcal{T}, \tau_{\text{semi}})$, and $(\mathcal{T}, \tau_{\text{tt}})$ are abbreviated as $\mathcal{T}_{\text{semi}}$, and $\mathcal{T}_{\text{tt}}$, respectively.

Example 2.39. Every $\mathbf{E}_{\text{ce}}$-point in the space $\mathcal{T}_{\text{semi}}$ is called a *c.e. trace*. Every $\mathbf{E}_{\text{ce}}$-point in the space $\mathcal{T}_{\text{tt}}$ is called a *computable trace*.

Definition 2.40. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are represented function spaces, $\tilde{\mathsf{Q}}$ is a collection of quantifiers, i.e., $\tilde{\mathsf{Q}} \subseteq \mathcal{PP}\omega$, and $\mathcal{S}$ is a represented space of slaloms. We say that *the $\mathcal{F}$-degree of a set $V \subseteq 2^\omega$ is $\tilde{\mathsf{Q}}$-often $\mathcal{S}$-traceable over* $\mathbf{E}$ (or V is $(\mathcal{S}, \tilde{\mathsf{Q}}; \mathcal{F}, \mathbf{E})$-traceable) if

$$(\forall f \in \mathcal{F}^{\mathbf{E}}(V, \omega^\omega))(\exists T \in \mathcal{S}^{\mathbf{E}})(\exists \mathsf{Q} \in \tilde{\mathsf{Q}})\ T\ \mathsf{Q}\text{-traces } f[V].$$

We also say that *the $\mathcal{F}$-degree of a set $V \subseteq 2^\omega$ is $\tilde{\mathsf{Q}}$-often $\mathcal{S}$-traceable $\mathcal{G}$-uniformly relative to $x \in 2^\omega$ over* $\mathbf{E}$ (or V is $(\mathcal{S}, \tilde{\mathsf{Q}}; \mathcal{F}, \mathcal{G}, \mathbf{E})$-traceable relative to x) if

$$(\forall f \in \mathcal{F}^{\mathbf{E}}(V, \omega^\omega))(\exists g \in \mathcal{G}^{\mathbf{E}}({}_{\subset}2^\omega, \mathcal{S}))(\exists \mathsf{Q} \in \tilde{\mathsf{Q}})\ g(x)\ \mathsf{Q}\text{-traces } f[V].$$

Example 2.41. The following are examples of traceability notions in algorithmic randomness theory (see [24, 37, 44, 57]).

(i) $(\mathcal{T}_{\rm semi}, \{\mathsf{Q}_{\rm inf}\}; \mathcal{C}, \mathbf{E}_{\rm ce})$-traceability is equal to c.e. i.o. traceablity.
(ii) $(\mathcal{T}_{\rm semi}, \{\mathsf{Q}_{\rm cof}\}; \mathcal{C}_{\rm tt}, \mathbf{E}_{\rm ce})$-traceability is equal to c.e. *tt*-traceablity.
(iii) $(\mathcal{T}_{\rm tt}, \{\mathsf{Q}_{\rm cof}\}; \mathcal{F}_{\mathbf{\Pi}^1_1}, \mathbf{E}_{\Pi^1_1})$-traceability is equal to Δ^1_1-traceablity.
(iv) $(\mathcal{T}_{\rm tt}, \{\mathsf{Q}_g : g \in (\omega^\omega)^{\mathbf{E}_{\rm ce}}\}; \mathcal{C}_{\rm tt}, \mathbf{E}_{\rm ce})$-traceability is equal to computably c.o. *tt*-traceablity.

Remark 2.42. Note that if $\tilde{\mathsf{Q}}$ is either $\{\mathsf{Q}_{\rm cof}\}$, $\{\mathsf{Q}_{\rm inf}\}$ or $\{\mathsf{Q}_g : g \in (\omega^\omega)^{\mathbf{E}}\}$, it is easy to see that the bound $|T_n| \leq 2^n$ in the definition of $\tilde{\mathsf{Q}}$-often traceability can be replaced with $|T_n| \leq p(n)$ for any unbounded nondecreasing $p \in (\omega^\omega)^{\mathbf{E}}$ (see [24, 57]).

2.5. **Kolmogorov Complexity.** It has been shown in the algorithmic randomness theory that most lowness notions are characterized in terms of Kolmogorov complexity [24, 57, 44]. One can introduce $\mathbf{E}$-relativized versions of Komogorov complexity by defining representations on the space $\mathcal{C}({}_{\subset}2^{<\omega}, 2^{<\omega})$ of machines as follows:

Definition 2.43 (Machine)**.** The space of *prefix-free machines* is a subspace $\mathcal{C}_{\rm pf}({}_{\subseteq}2^{<\omega}, 2^{<\omega})$ of the function space $\mathcal{C}({}_{\subseteq}2^{<\omega}, 2^{<\omega})$ consisting of functions $\varphi :\subseteq 2^{<\omega} \to 2^{<\omega}$ such that $\mathrm{dom}(\varphi)$ is prefix-free. Note that $\mathcal{C}({}_{\subseteq}2^{<\omega}, 2^{<\omega})$ has the induced representation by Definitions 2.11 and 2.14 via an effective representation of the discrete space $2^{<\omega} \simeq \omega$. The representation $\rho_{\rm semi}$ of $\mathcal{C}_{\rm pf}({}_{\subseteq}2^{<\omega}, 2^{<\omega})$ is obtained by restricting its range to prefix-free functions. In other words,

$$\rho_{\rm semi}(p) = \varphi \iff p = \mathrm{Graph}(\varphi), \text{ where } \mathrm{dom}(\rho_{\rm semi}) = \{p : \rho_{\rm semi}(p) \in \mathcal{C}_{\rm pf}(2^{<\omega}, 2^{<\omega})\}.$$

Moreover, we also consider the following representation $\rho_{\rm tt\lambda} :\subseteq \mathcal{O}\omega \to \mathcal{C}_{\rm pf}(2^{<\omega}, 2^{<\omega})$:

$$\rho_{\rm tt\lambda}(\langle p, q\rangle) = \rho_{\rm semi}(p), \text{ where } \mathrm{dom}(\rho_{\rm tt\lambda}) = \{\langle p, q\rangle : \rho_{\rm semi}(p) \in \mathcal{C}_{\rm pf}(2^{<\omega}, 2^{<\omega}), \text{ and } \rho_{\mathbb{R}}(q) = \Omega_{\rho_{\rm semi}(p)}\}.$$

Here, the halting probability Ω_φ of a machine is defined as $\Omega_\varphi = \lambda([\![\mathrm{dom}(\varphi)]\!]) = \sum_{\sigma \in \mathrm{dom}(\varphi)} 2^{-|\sigma|}$, where λ is the Lebesgue measure on 2^ω. Then, the represented spaces $(\mathcal{C}_{\rm pf}(2^{<\omega}, 2^{<\omega}), \rho_e)$ and $(\mathcal{C}_{\rm pf}(2^{<\omega}, 2^{<\omega}), \rho_{\rm tt\lambda})$ are abbreviated as $\mathcal{C}_{\rm semi}(2^{<\omega})$ and $\mathcal{C}_{\rm tt\lambda}(2^{<\omega})$, respectively.

Example 2.44. Every $\mathbf{E}_{\rm ce}$-point in the space $\mathcal{C}_{\rm semi}(2^{<\omega})$ is exactly a partial computable prefix-free machine. Every $\mathbf{E}_{\rm ce}$-point in the space $\mathcal{C}_{\rm tt\lambda}(2^{<\omega})$ is called a *computable measure machine*, that is, its halting probability is computable (see [24, Section 7.1.3]).

The *prefix-free Kolmogorov complexity K_φ with respect to a machine* φ is defined by $K_\varphi(\sigma) = \min\{|\tau| : \varphi(\tau) = \sigma\}$. We define $\mathbf{E}$-relativized versions of K-triviality, Schnorr triviality, and K-reducibility as follows.

Definition 2.45. Let $r \in \{\mathrm{semi}, \mathrm{tt}\lambda\}$. A set $A \subseteq \omega$ *is K_r-reducible to $B \subseteq \omega$ over* $\mathbf{E}$ (written as $A \leq^{\mathbf{E}}_{K_r} B$) if

$$(\forall \varphi \in \mathcal{C}^{\mathbf{M}}_r(2^{<\omega}))(\exists \psi \in \mathcal{C}^{\mathbf{M}}_r(2^{<\omega}))\ K_\psi(A \upharpoonright n) \leq K_\varphi(B \upharpoonright n) + O(1).$$

We write $A \equiv^M_{K_r} B$ if $A \leq^M_{K_r} B$ and $B \leq^M_{K_r} A$. We also simply replace K_r with K if $r = $ semi. A set $A \subseteq \mathbb{N}$ is said to be *K_r-trivial over* $\mathbf{E}$ if $A \leq^{\mathbf{E}}_{K_r} \emptyset$.

Example 2.46. If a set $A \subseteq \omega$ is $K_{\rm semi}$-trivial over $\mathbf{E}_{\rm ce}$, it is usually called K-trivial. If $A \subseteq \omega$ is $K_{\rm tt\lambda}$-trivial over $\mathbf{E}_{\rm ce}$, it is usually called Schnorr trivial.

Definition 2.47. We say that a set $A \subseteq \omega$ is $\tilde{\mathsf{Q}}$*-often* K_r*-compressible over* $\mathbf{E}$ if

$$(\forall g \in (\omega^\omega)^{\mathbf{E}})(\varphi \in \mathcal{C}_r(2^{<\omega}))(\exists \mathsf{Q} \in \tilde{\mathsf{Q}})\ \{n \in \omega : K_\varphi(A \upharpoonright g(n)) \leq n\} \in \mathsf{Q}.$$

We also say that a set $A \subseteq \omega$ is $\tilde{\mathsf{Q}}$*-often* K_r*-autocompressible over* $\mathbf{E}$ if

$$(\forall g \leq^{\mathbf{E}}_{\mathcal{C}} A)(\varphi \in \mathcal{C}_r(2^{<\omega}))(\exists \mathsf{Q} \in \tilde{\mathsf{Q}})\ \{n \in \omega : K_\varphi(A \upharpoonright g(n)) \leq n\} \in \mathsf{Q}.$$

Example 2.48. The following are examples of complexity properties in algorithmic randomness theory (see [24, 37, 44]).

(i) A is complex iff A is not $\{\mathsf{Q}_{\rm inf}\}$-often $K_{\rm semi}$-compressible over $\mathbf{E}_{\rm ce}$.
(ii) A is autocomplex iff A is not $\{\mathsf{Q}_{\rm inf}\}$-often $K_{\rm semi}$-autocompressible over $\mathbf{E}_{\rm ce}$.
(iii) A is anticomplex iff A is $\{\mathsf{Q}_{\rm cof}\}$-often $K_{\rm semi}$-compressible over $\mathbf{E}_{\rm ce}$.
(iv) A is totally complex iff A is not $\{\mathsf{Q}_{\rm inf}\}$-often $K_{{\rm tt}\lambda}$-compressible over $\mathbf{E}_{\rm ce}$.

It is known that $(\mathcal{T}_s, \tilde{\mathsf{Q}}; \mathcal{C}_{\rm tt}, \mathbf{E}_{\rm ce})$-traceability is equivalent to $\tilde{\mathsf{Q}}$-often K_r-compressibility over $\mathbf{E}_{\rm ce}$, where $s = {\rm semi}$ iff $r = {\rm semi}$; and $s = {\rm tt}$ iff $r = {\rm tt}\lambda$ (see [24, 37, 44]). It is also known that $(\mathcal{T}_s, \tilde{\mathsf{Q}}; \mathcal{C}, \mathbf{E}_{\rm ce})$-traceability is equivalent to $\tilde{\mathsf{Q}}$-often K_r-autocompressibility over $\mathbf{E}_{\rm ce}$. One can easily generalize the first equivalence.

Proposition 2.49 (see [37])**.** *Suppose that* $s = r = {\rm semi}$*, or* $s = {\rm tt}$ *and* $r = {\rm tt}\lambda$*. Then,* $(\mathcal{T}_s, \tilde{\mathsf{Q}}; \mathcal{C}_{\rm tt}, \mathbf{E})$*-traceability is equivalent to* $\tilde{\mathsf{Q}}$*-often* K_r*-compressibility over* $\mathbf{E}$*.* □

3. Basic Equivalences

3.1. **Additivity, Lowness and Preservation.** We first show that $(\mathcal{I}, \mathcal{J})$-additivity is equivalent to $(\mathcal{I}, \mathcal{J})$-randomness preserving for any well-behaved pair $(\mathcal{I}, \mathcal{J})$. Inspired by Rute's terminology [65], we say that a measure-preserving map T satisfies *randomness-preservation w.r.t.* $(\mathcal{I}, \mathcal{J})$ if $T(x)$ is $\mathcal{J}$-random whenever x is $\mathcal{I}$-random, and that T satisfies *no-randomness-from-nothing w.r.t.* $(\mathcal{I}, \mathcal{J})$ if $T(x)$ is not $\mathcal{J}$-random whenever x is not $\mathcal{I}$-random. Now, we consider the measure-preserving map $T_A : 2^\omega \to 2^\omega$ defined by $Z \mapsto A \triangle Z$. Note that, since $A \triangle Z \triangle Z = A$, one can easily see that T_A satisfies randomness-preservation w.r.t. $(\mathcal{I}, \mathcal{J})$ if and only if T_A satisfies no-randomness-from-nothing w.r.t. $(\mathcal{J}, \mathcal{I})$.

Definition 3.1. A set $A \subseteq \omega$ is $(\mathcal{I}, \mathcal{J})$*-randomness preserving over* $\mathbf{E}$ if T_A satisfies randomness-preservation w.r.t. $(\mathcal{I}, \mathcal{J})$ over $\mathbf{E}$, that is, $A \triangle Z$ is $\mathcal{J}$-random whenever $Z \subseteq \omega$ is $\mathcal{I}$-random.

We now compare the notion of transitive additivity and randomness preservation. Consider the following sets G_0^-, G_1^- of pairs of ideals.

$$G_0^- = \{(\mathcal{E}_{\rm WR}, \mathcal{E}_{\rm WR}), (\mathcal{M}_{\rm W1G}, \mathcal{M}_{\rm W1G})\}, \quad G_1^- = \{(\mathcal{N}_{\rm MLR}, \mathcal{N}_{\rm SR}), (\mathcal{N}_{\rm MLR}, \mathcal{E}_{\rm WR})\}.$$

We say that $(\mathcal{I}, \mathcal{J})$ is a *good*$_i^-$-pair of σ-ideals if $(\mathcal{I}, \mathcal{J}) \in \bigcup_{j \leq i} G_j^-$. We will see the following:

Theorem 3.2. *Suppose that* $\mathbf{E}$ *is a countable locally KS set, and* $(\mathcal{I}, \mathcal{J})$ *is a good*$_0^-$*-pair of* σ*-ideals. Then, a set* $A \subseteq \omega$ *is* $(\mathcal{J}, \mathcal{I})$*-additive over* $\mathbf{E}$ *if and only if* A *is* $(\mathcal{I}, \mathcal{J})$*-randomness preserving over* $\mathbf{E}$*. If* $\mathbf{E}$ *is principal KS, then* $(\mathcal{I}, \mathcal{J})$ *may be chosen as a good*$_1^-$*-pair of* σ*-ideals.*

Lemma 3.3. *If a set* $A \subseteq \omega$ *is* $(\mathcal{J}, \mathcal{I})$*-additive over* $\mathbf{E}$*, then* A *is* $(\mathcal{I}, \mathcal{J})$*-randomness preserving over* $\mathbf{E}$*.*

Proof. Assume that $A \subseteq \mathbb{N}$ is $(\mathcal{J}, \mathcal{I})$-additive over $\mathbf{E}$. It suffices to show that T_A satisfies no-randomness-from-nothing w.r.t. $(\mathcal{J}, \mathcal{I})$ over $\mathbf{E}$. Let $Z \subseteq \omega$ be a set such that $\{Z\} \in \mathcal{J}^{\mathbf{E}}$. Then $\{A\} + \{Z\} = \{A \triangle Z\} \in \mathcal{I}^{\mathbf{E}}$ by $(\mathcal{J}, \mathcal{I})$-additivity of A over $\mathbf{E}$. □

Recall that a *universal* $\mathcal{J}^{\mathbf{E}}$*-test* is a greatest element in $\mathcal{J}^{\mathbf{E}}$. For instance, $(\mathcal{N}, \rho_{\rm MLR})^{\mathbf{E}_{\rm ce}}$ has a greatest element called a *universal Martin-Löf test*. The ideal $(\mathcal{M}, \rho_{\rm 1G})^{\mathbf{E}_{\rm ce}}$ also has a greatest element.

Lemma 3.4. *Assume that* $\mathcal{J}^{\mathbf{E}}$ *has a universal test. If a set* $A \subseteq \omega$ *is* $(\mathcal{J}, \mathcal{I})$*-randomness preserving over* $\mathbf{E}$*, then* A *is* $(\mathcal{I}, \mathcal{J})$*-additive over* $\mathbf{E}$*.*

Proof. Let $U \in \mathcal{J}^{\mathbf{E}}$ be a universal test. Clearly, if a real is not contained in a universal $\mathcal{J}^{\mathbf{E}}$-test, then it must be a $\mathcal{J}$-generic over $\mathbf{E}$. Suppose that A is not $(\mathcal{I}, \mathcal{J})$-additive over $\mathbf{E}$. Then, there is $N \in \mathcal{I}$ such that $\{A\} + N \notin \mathcal{J}$. Hence, $\{A\} + N \not\subseteq U$. Fix $Z \in \{A\} + N \setminus U$. Then, Z is $\mathcal{J}$-generic over $\mathbf{E}$ since $Z \notin U$. Moreover, $A \triangle Z \in N \in \mathcal{I}$ since $Z \in \{A\} + N$. Hence, Z is not $\mathcal{I}$-generic over $\mathbf{E}$. □

We have another condition ensuring that the above property holds. Call $H \subseteq 2^\omega$ *infinitely often (i.o.) homogeneous* if for infinitely many n and for any $\sigma, \tau \in 2^n$, if $H \cap [\sigma]$ and $H \cap [\tau]$ are nonempty, then these sets are equivalent above level n. Such an n is called a homogeneity level for H. We say that a subclass $\mathcal{I}^{\mathbf{E}}$ of an ideal $\mathcal{I}$ is *$\mathcal{M}$-like* if it is generated in $\mathbf{E}$ by a class $\mathcal{P} \subseteq (\mathbf{\Pi}^0_1(2^\omega), \pi^0_1)^{\mathbf{E}}$ of $\mathbf{E}$-semicoded closed sets (see Definition 2.3) such that every $N \in \mathcal{P}$ is included in an i.o. homogeneous set $N^* \in \mathcal{P}$.

Example 3.5. $(\mathcal{M}, \rho_{\mathrm{W1G}})$ and $(\mathcal{E}, \rho_{\mathrm{WR}})$ are $\mathcal{M}$-like (see also Section 4.4).

Let $\mathcal{I}$ be a class of subsets of 2^ω. The class $\mathcal{I}$ is said to be *closed under finite duplication* provided $N \in \mathcal{I}$ implies $\mathrm{dup}_{|\sigma|}(N \cap [\sigma]) \in \mathcal{I}$ for every string σ. Here,

$$\mathrm{dup}_{|\sigma|}(N \cap [\sigma]) = \{\tau ^\frown g : \tau \in 2^{|\sigma|} \text{ and } \sigma ^\frown g \in N\}.$$

Lemma 3.6. *Assume that $\mathcal{I}$ is $\mathcal{M}$-like, and $\mathcal{J}$ is a countable class generated by closed sets that are closed under finite duplication. If a set $A \subseteq \omega$ is $(\mathcal{J}, \mathcal{I})$-randomness preserving over $\mathbf{E}$, then A is $(\mathcal{I}, \mathcal{J})$-additive over $\mathbf{E}$.*

Lemma 3.7. *Let $\mathcal{I}$ be a countable class generated by closed sets that are closed under finite duplication. For every i.o. homogeneous closed set $H \subseteq 2^\omega$, if H is not covered by any $N \in \mathcal{I}$, then H is not covered by $\bigcup \mathcal{I}$.* □

Proof of Lemma 3.6. Suppose for the sake of contradiction that A is not $(\mathcal{I}, \mathcal{J})$-additive. Then, there is $N \in \mathcal{I}$ such that $\{A\} + N \notin \mathcal{J}$. Note that we may assume that N is i.o. homogeneous, since $\mathcal{I}$ is $\mathcal{M}$-like. It is easy to see that $\{A\} + N$ is also i.o. homogeneous. Then, by Lemma 3.7, $\{A\} + N$ is not covered by $\bigcup \mathcal{J}$. Fix $Z \in \{A\} + N \setminus \bigcup \mathcal{J}$. Then, Z is $\mathcal{J}$-generic over $\mathbf{E}$ since $Z \notin \bigcup \mathcal{J}$. Moreover, $A \triangle Z \in N \in \mathcal{I}$ since $Z \in \{A\} + N$. Hence, Z is not $\mathcal{I}$-generic over $\mathbf{E}$. □

Proof of Theorem 3.2. We need countability of $\mathbf{E}$ to use Lemma 3.6. If $\mathbf{E}$ is principal, $\mathbf{E} = \mathbf{E}_\Gamma$ for an ω-parametrized pointclass Γ. Therefore, there is a universal $\mathbf{E}$-Martin-Löf test. Hence, we can use Lemma 3.4. □

As a consequence, for instance, Martin-Löf null-additivity is equivalent to Martin-Löf random-preservation, Kurtz null-additivity is equivalent to Kurtz random-preservation, etc.

Remark 3.8. Given a property $\mathcal{P}$ for a subset of ω, a set $A \subseteq \omega$ is said to be *hereditary* $\mathcal{P}$ if B satisfies $\mathcal{P}$ for every $B \subseteq A$. It is not hard to show the existence of a hereditary Schnorr random preserving set (see [28]). Let $\mathcal{C}$ and $\mathcal{D}$ be collections of subsets of ω. A set $A \subseteq \omega$ is called *universally indifferent w.r.t.* $(\mathcal{C}, \mathcal{D})$ if $Z \triangle B \in \mathcal{D}$ for any $Z \in \mathcal{C}$ and $B \subseteq A$ (see [20, 26]). A set $A \subseteq \omega$ is hereditary $(\mathcal{I}, \mathcal{J})$-randomness preserving over $\mathbf{E}$ if and only if it is universally indifferent w.r.t. $(\mathsf{RND}^{\mathbf{E}}_{\mathcal{I}}, \mathsf{RND}^{\mathbf{E}}_{\mathcal{J}})$. However, it is known that there is no set universally indifferent w.r.t. Martin-Löf randomness (see [26]). Therefore, there is no hereditary Martin-Löf null-additive set.

3.2. **Lowness versus Additivity.** We next see that the notion of $\mathrm{low}_{\mathrm{test}}$-ness is essentially equivalent to the notion of smallness in the context of additivity numbers (see Definition 2.34). Later, we will also see that $\mathrm{low}_{\mathrm{test}}$-ness w.r.t. uniform relativization (i.e., $\mathcal{C}_{\mathrm{tt}}$-relativization) is essentially equivalent to transitive additivity (see Definition 2.36).

Proposition 3.9. *Let $\mathcal{I}$ and $\mathcal{J}$ be ideals, $\mathcal{F}$ be any represented function space, $\mathbf{E}$ be any set. Then, $(\mathcal{I}, \mathcal{J}; \mathcal{F}, \mathbf{E})$-$\mathrm{low}_{\mathrm{test}}$-ness is equal to $(\mathcal{J}, \mathcal{I}; \mathcal{F}, \mathbf{E})$-smallness.*

Proof. Note that $\{z\} \in \mathrm{Add}^{\mathcal{F}}_{\mathbf{E}}(\mathcal{J}, \mathcal{I})$ if and only if $J(z) \in \mathcal{I}^{\mathbf{E}}$ for every $\mathcal{F}^{\mathbf{E}}$-function $J : A \to \mathcal{J}$. Clearly, this condition is equivalent to $\mathcal{F}$-uniform lowness for $(\mathcal{I}, \mathcal{J})$-tests over $\mathbf{E}$. □

We then discuss the relationship between $(\mathcal{I}, \mathcal{J}; \mathcal{F}, \mathbf{E})$-lowness and $(\mathcal{I}, \mathcal{J}; \mathcal{F}, \mathbf{E})$-$\mathrm{low}_{\mathrm{test}}$-ness. Consider the following sets G_0, G_1, G_2 of pairs of ideals.

$$G_0 = \{(\mathcal{N}_{\mathrm{SR}}, \mathcal{N}_{\mathrm{SR}}), (\mathcal{E}_{\mathrm{WR}}, \mathcal{E}_{\mathrm{WR}}), (\mathcal{M}_{\mathrm{W1G}}, \mathcal{M}_{\mathrm{W1G}})\},$$
$$G_1 = \{(\mathcal{N}_{\mathrm{MLR}}, \mathcal{N}_{\mathrm{SR}}), (\mathcal{N}_{\mathrm{MLR}}, \mathcal{E}_{\mathrm{WR}})\}, \quad G_2 = \{(\mathcal{N}_{\mathrm{SR}}, \mathcal{N}_{\mathrm{WR}})\}.$$

We say that $(\mathcal{I}, \mathcal{J})$ is a *good$_i$*-pair of σ-ideals if $(\mathcal{I}, \mathcal{J}) \in \bigcup_{j \le i} G_j$. We also say that a represented function space $\mathcal{F}$ is $\mathbf{E}$-*good* if it is closed under taking composition, and the inclusion map $\mathcal{C}_{\mathrm{tt}} \hookrightarrow \mathcal{F}$ is $\mathbf{E}$-semicoded.

Theorem 3.10. *Suppose that* $\mathbf{E}$ *is a countable locally KS set,* $\mathcal{F}$ *is an* $\mathbf{E}$*-good function space, and* $(\mathcal{I},\mathcal{J})$ *is a good*$_0$*-pair of* σ*-ideals. Then,* $(\mathcal{I},\mathcal{J};\mathcal{F},\mathbf{E})$*-lowness is equal to* $(\mathcal{I},\mathcal{J};\mathcal{F},\mathbf{E})$*-low*$_{\rm test}$*-ness. If* $\mathbf{E}$ *is principal KS, then* $(\mathcal{I},\mathcal{J})$ *may be chosen as a good*$_1$*-pair of* σ*-ideals.*

Proof (Sketch). It is easy to see that $(\mathcal{I},\mathcal{J};\mathcal{F},\mathbf{E})$-low$_{\rm test}$-ness implies $(\mathcal{I},\mathcal{J};\mathcal{F},\mathbf{E})$-lowness. If $\mathcal{I},\mathcal{J}\in\{\mathcal{N}_{\rm MLR},\mathcal{N}_{\rm SR}\}$, then one can adopt the argument by Bienvenu-Miller [6, Section 4.3]. Here, we need countability of $\mathbf{E}$ to ensure that there are only countably many tests. Then, we can use [6, Lemma 12] to show the equivalence between lowness and low$_{\rm test}$-ness for $\mathcal{N}_{\rm SR}$ w.r.t. $\mathcal{F}$-uniformization over $\mathbf{E}$. Moreover, we need principality of $\mathbf{E}$ to ensure the existence of a universal Martin-Löf test, which is used to show [6, Theorem 8]. If $\mathcal{I}$ has a universal test, then it is easy to show the equivalence between lowness and low$_{\rm test}$-ness for $(\mathcal{I},\mathcal{J})$ w.r.t. $\mathcal{F}$-uniformization over $\mathbf{E}$. If $\mathcal{I},\mathcal{J}\in\{\mathcal{E}_{\rm WR},\mathcal{M}_{\rm W1G}\}$, since $\mathcal{I}$ and $\mathcal{J}$ are $\mathcal{M}$-like, use Lemma 3.7. □

One can also see the relationship between the low-for-randomness (*LR*) degrees (see [57]) and some cardinal characteristics (see [3]).

Proposition 3.11. *The cardinal characteristic* $\mathsf{add}(\mathcal{N})$ *is equal to the least cardinality of* $X\subseteq 2^\omega$ *whose LR-degrees is unbounded in the whole LR-degrees. The cardinal characteristic* $\mathsf{cof}(\mathcal{N})$ *is equal to the least cardinality of* $X\subseteq 2^\omega$ *whose LR-degrees is cofinal in the whole LR-degrees.*

Proof. Note that if ρ is a representation of $\mathcal{N}$, then the existence of a collection $\{N_x\}_{x\in X}$ of null sets with $|X|=\kappa$ such that $\bigcup_{x\in X}N_x$ is not null is equivalent to the existence of a collection Y of cardinality κ such that $\bigcup_{y\in Y}\rho(y)\not\subseteq\rho(z)$ for any $z\in{\rm dom}(\rho)$. Fix a total representation ρ such that $\rho(z)$ is a universal Martin-Löf test relative to z. Then, $\bigcup_{y\in Y}\rho(y)\not\subseteq\rho(z)$ for any $z\in 2^\omega$ implies that for every $z\in 2^\omega$, there is $r\in\bigcup_{y\in Y}\rho(y)$ such that $r\notin\rho(z)$, which means that r is not Martin-Löf random relative to y for some $y\in Y$, but r is Martin-Löf random relative to z. In other words, for every $z\in 2^\omega$, there is $y\in Y$ such that y is not LR-reducible to z.

For the second assertion, the existence of a collection $\{N_x\}_{x\in X}$ of null sets with $|X|=\kappa$ such that every null set is covered by N_x for some $x\in X$ is equivalent to the existence of a collection Y of cardinality κ such that for every $z\in{\rm dom}(\rho)$, there is $y\in Y$ such that $\rho(z)\subseteq\rho(y)$. Then, $\rho(z)\subseteq\rho(y)$ is clearly equivalent to $z\leq_{\rm LR}y$. □

3.3. **Traceability and Smallness.** In this subsection, we show a technical lemma converting tt-traceability into "$\star$-traceability" which can be viewed as a certain kind of measure-theoretic smallness.

A sequence $\mathcal{I}$ of finite intervals $I_n\subseteq\omega$ generates the total continuous function $c_{\mathcal{I}}:2^\omega\to\omega^\omega$ such that $c_{\mathcal{I}}(x)(n)=x\upharpoonright I_n$, where the finite string $x\upharpoonright I_n$ is identified with a natural number via a fixed effective bijecton. Let $\omega^{\uparrow\omega}$ denote the set of all increasing functions. Every $u\in\omega^{\uparrow\omega}$ generates sequences $\mathcal{I}[u]:=([0,u(n)])_{n\in\omega}$ and $\mathcal{J}[u]:=((u(n-1),u(n)])_{n\in\omega}$, where $u(-1)=-1$. Let $\mathcal{C}^\star_{\rm tt}$ denote the space of total continuous functions of the form $c_{\mathcal{I}[u]}$ for some $u\in\omega^{\uparrow\omega}$, where each function $c_{\mathcal{I}[u]}$ is represented by a name of $u\in\omega^{\uparrow\omega}$. Then we sometimes use the term $\tilde{\mathsf{Q}}$*-often* $\mathcal{S}$*-*$\star$*-traceability over* $\mathbf{M}$ for $(\mathcal{S},\tilde{\mathsf{Q}};\mathcal{C}^\star_{\rm tt},\mathbf{M})$-traceability.

Lemma 3.12. *Suppose that* $\mathbf{E}$ *is a locally KS set, and* $\mathcal{G}$ *is an* $\mathbf{E}$*-good function space (see Section 3.2). Then, a set* $V\subseteq 2^\omega$ *is* $(\mathcal{T}_{\rm tt},\tilde{\mathsf{Q}};\mathcal{C}^\star_{\rm tt},\mathcal{G},\mathbf{E})$*-traceable relative to* x *if and only if it is* $(\mathcal{T}_{\rm tt},\tilde{\mathsf{Q}};\mathcal{C}_{\rm tt},\mathcal{G},\mathbf{E})$*-traceable relative to* x*.*

Proof. One direction is easy since $A\mapsto A^{\upharpoonright u}$ is contained in $\mathcal{C}^{\mathbf{E}}_{\rm tt}$ for every $u\in\mathbf{E}$. For another direction, suppose that $V\subseteq 2^\omega$ is $\mathcal{C}_{\rm tt}$-traceable $\mathcal{G}$-uniformly in x over $\mathbf{E}$, and fix $f\in\mathcal{C}^{\mathbf{E}}_{\rm tt}$. Then, by Lemma 2.10, one can effectively find a modulus of continuity $u\in\mathbf{E}$ of f, that is, $f(z)(n)$ is determined by $z\upharpoonright u(n)$ for every $n\in\omega$. For every trace T, we consider

$$F(T)=\{f(\sigma)(n):\sigma\in T(n)\ \&\ |\sigma|=u(n)\}.$$

Then, $F(T)$ is also a trace. Moreover, it is not hard to see that $T\mapsto F(T)$ is contained in $\mathcal{C}^{\mathbf{E}}_{\rm tt}(\mathcal{T}_{\rm semi},\mathcal{T}_{\rm semi})$ and $\mathcal{C}^{\mathbf{E}}_{\rm tt}(\mathcal{T}_{\rm tt},\mathcal{T}_{\rm tt})$. Therefore, for any $\mathcal{C}^{\mathbf{E}}_{\rm tt}$-uniform (semi-)$\star$-trace g, the function $h=F\circ g$ is also a $\mathcal{C}^{\mathbf{E}}_{\rm tt}$-uniform $\star$-trace. Moreover, if g traces $V^{\upharpoonright u}$, then h clearly traces $f[V]$. □

Remark 3.13. The notion of $\star$-traceability (i.o. $\star$-traceability, resp.) over $\mathbf{E}$ is equivalent to being measure zero with respect to h-dimensional packing outer measure (Hausforff outer measure, resp.) for any gauge function $h\in\mathbf{E}$ (see also [44, 62, 79]). By Proposition 2.49 and Lemma 3.12, every $\star$-traceability notion is characterized by using the Kolmogorov complexity (see also [37, 44]).

Note that we can replace the above $\star$-traceability notions by the following piecewise traceability notions. We say that $x \in 2^\omega$ Q-*often* $\mathcal{I}$-*traces* $V \subseteq 2^\omega$ if the slalom $(\{c_{\mathcal{I}}(x)(n)\})_{n\in\omega}$ Q-often traces V. If $\mathcal{I}$ is of the form $\mathcal{I}[u]$ ($\mathcal{J}[u]$, resp.) for some $u \in \omega^{\uparrow\omega}$, then we say that x Q-*often* u-*traces* (Q-*often piecewise* u-*traces*, resp.) V.

Lemma 3.14. *Suppose that* $\mathbf{E}$ *is a locally KS set. For any order* u*, an* $\mathbf{E}$*-coded real* Q*-often* u*-traces* X *if and only if for any order* u*, an* $\mathbf{E}$*-coded real* Q*-often piecewise* u*-traces* X*.*

Proof. If φ u-traces X, then clearly it piecewise u-traces X. Let u be a given order. Inductively define another order v. Put $v(0) = u(0)$ and assume that $v(n)$ has been defined to be $u(k_n)$ for some $k_n \in \omega$. Then, put $v(n+1) = u(k_n + 2^{v(n)})$. Assume that piecewise φ v-traces X. Let $\{\sigma^n_s\}_{s<2^{v(n)}}$ be a list of all strings of length $v(n)$. Then for each $n \in \omega$ and $s < 2^{v(n)}$, define $\psi(0) = \varphi(0)$ and $\psi(k_n + s) = \sigma^n_s{}^\frown\varphi(n+1)\upharpoonright u(k_n + s)$. If φ piecewise v-traces X, then ψ u-traces X. $\square$

3.4. **Triviality and Complexity Preservation.** Note that Schnorr triviality implies computable *tt*-traceability [28], and it implies uniform lowness for computable measure machine [52]. Clearly, uniform lowness for computable measure machine implies that $A\triangle Z \equiv_{\rm Sch} Z$ for every $Z \subseteq \mathbb{N}$, and it implies Schnorr-triviality. Consequently, a set $A \subseteq \omega$ *preserves Kolmogorov complexity w.r.t. computable measure machines* if and only if A is Schnorr trivial. Generally, we say that a set $A \subseteq \omega$ is K_r*-complexity preserving over* $\mathbf{E}$ if $A\triangle Z \equiv^{\mathbf{E}}_{K_r} Z$ for every $Z \subseteq \omega$.

Theorem 3.15. *Let* $\mathbf{E} \subseteq \mathcal{P}\omega$ *be a locally KS set. The following are pairwise equivalent for a set* $A \subseteq \mathbb{N}$*.*

- (i) A *is* $K_{\rm tt\lambda}$*-trivial over* $\mathbf{E}$*.*
- (ii) A *is* $K_{\rm tt\lambda}$*-complexity preserving over* $\mathbf{E}$*.*
- (iii) *The singleton* $\{A\}$ *is* $\mathcal{T}_{\rm tt}$*-*$\star$*-traceable over* $\mathbf{E}$*.*

Proof. (ii)⇒(i): By the definition of $K_{\rm tt\lambda}$-complexity preserving, we have $A = A\triangle\emptyset \equiv^{\mathbf{E}}_{K_{\rm tt\lambda}} \emptyset$. This is equivalent to that A is $K_{\rm tt\lambda}$-trivial over $\mathbf{E}$.

(i)⇒(iii): Assume that A is $K_{\rm tt\lambda}$-trivial over $\mathbf{E}$. Let $(\mathbf{E}_a)_{a\in\Lambda}$ be a KS base of $\mathbf{E}$. Recall that $\mathbf{E}_a$ is an e-ideal, i.e., it is downward closed under enumeration reducibility $\leq_{\rm e}$ and join $\oplus$, for every $a \in \Lambda$, since any Σ-pointclass Γ_a includes the pointclass Σ^0_1 by the definition (see Moschovakis [55]). Let $u \in \omega^\omega$ be any order coded in $\mathbf{E}$, say $\mathbf{E}_a$-coded. We define a machine $\varphi \in \mathcal{C}_{\rm tt\lambda}(2^{<\omega})$ by $\varphi(1^n0) = u(n)$ for every n. Clearly φ is $\mathbf{E}_a$-coded since $\mathbf{E}_a$ is an e-ideal. By $K_{\rm tt\lambda}$-triviality of A over $\mathbf{E}$, there exists an $\mathbf{E}$-coded (say, $\mathbf{E}_b$-coded) machine $\psi \in \mathcal{C}^{\mathbf{E}}_{\rm tt\lambda}$ such that $K_\psi(A\upharpoonright u(n)) \leq K_\varphi(u(n)) + O(1) \leq n + O(1)$. Now, $T_n = \{\sigma \in 2^{<\omega} : K_\psi(\sigma) < 2n\}$ is computable relative to a code of ψ (which has an information on its halting probability Ω_ψ) uniformly in n. Moreover, $A\upharpoonright u(n) \in T_n$ and $|T_n| < 2^{2n}$. Then, $(T_n)_{n\in\omega} \in \mathcal{T}_{\rm tt}$ is $\mathbf{E}_b$-coded since $\mathbf{E}_b$ is an e-ideal. Since u is arbitrary, $\{A\}$ is $\mathcal{T}_{\rm tt}$-$\star$-traceable over $\mathbf{E}$.

(iii)⇒(ii): Let $\varphi \in \mathcal{C}^{\mathbf{E}}_{\rm tt\lambda}(2^{<\omega})$ be an $\mathbf{E}$-coded (say, $\mathbf{E}_a$-coded) machine. By definition, the following sets $S_0, S_1 \in \mathcal{P}\omega$ is in Γ_a (via a trivial effective identifications among ω, $2^{<\omega}$, $\mathbb{Q}$, etc.):

$$S_0 := \{(\sigma,\tau) \in 2^{<\omega} \times 2^{<\omega} : \varphi(\sigma) = \tau\}, \qquad S_1 := \{(q,r) \in \mathbb{Q}\times\omega : |\Omega_\varphi - q| < 2^{-r}\}.$$

Given a finite set $F \subseteq S_0$ we denote $\Omega_F = \sum_{(\sigma,\tau)\in F} 2^{-|\sigma|}$. For any n, there are a finite set $F \subseteq S_0$ and $(q,r) \in S_1$ such that $r > 2n+2$ and $|\Omega_F - q| < 2^{-r}$. By uniformization, we have an $\mathbf{E}_a$-coded sequence $\langle\varphi_n\rangle_{n\in\omega}$ of finite functions approximating (the graph of) φ such that $\Omega_\varphi - \Omega_{\varphi_s} < 2^{-2n-1}$. Put $u(n) = \max\{|\sigma| : \sigma \in {\rm dom}(\varphi_n)\}$. Clearly, u is $\mathbf{E}_a$-coded. Since $\{A\}$ is $\mathcal{T}_{\rm tt}$-$\star$-traceable over $\mathbf{E}$, there exists an $\mathbf{E}_b$-coded sequence $\{T_n\}_{n\in\omega}$ of sets $T_n \subseteq 2^{u(n)}$ such that $|T_n| \leq 2^n$ and $A\upharpoonright u(n) \in T_n$. Let $\mathbf{E}_c$ be an amalgamation of $\mathbf{E}_a$ and $\mathbf{E}_b$. Then, the weight of the request set

$$L = \bigcup_n \{\langle\sigma\triangle\tau\upharpoonright|\sigma|, K_{\varphi_n}(\sigma)\rangle : \sigma \in {\rm dom}(\varphi_{n+1}) \setminus {\rm dom}(\varphi_n) \text{ and } \tau \in T_n\}.$$

is less than $\sum_n 2^{-2n-1}\cdot|T_n| \leq 1$, and its exact value is computable in $(\varphi_n)_{n\in\omega}\oplus(T_n)_{n\in\omega}$, and hence $\mathbf{E}_c$-coded. Moreover, L itself is $\mathbf{E}_c$-decidable. Hence, by the standard KC set argument, we have an L-computable (in particular, $\mathbf{E}_c$-coded) machine $\psi \in \mathcal{C}^{\mathbf{E}_c}_{\rm tt\lambda}$ such that $K_\psi(\sigma\triangle A\upharpoonright|\sigma|) = K_\varphi(\sigma)$ for every $\sigma \in {\rm dom}(\varphi)\setminus{\rm dom}(\varphi_0)$. Since ${\rm dom}(\varphi_0)$ is finite, we have $A\triangle Z \leq^{\mathbf{E}}_K Z$ for every $Z \subseteq \mathbb{N}$. Moreover, for every $Y \subseteq \mathbb{N}$, by putting $Z = A\triangle Y$, we also have $Y = A\triangle(A\triangle Y) \leq^{\mathbf{E}}_K A\triangle Y$. $\square$

4. Transitive Additivity versus Uniform Relativization

4.1. **Additivity, Triviality and Uniform Tests.** The purpose of this section is to show the equivalence between transitive additivity and lowness for randomness w.r.t. $\mathcal{C}_{\rm tt}$-relativization. By combining with Lemmata 3.4, 3.6 and 3.7, our results in this section imply Theorem 1.1 in introduction.

For (v) and (vi) of Theorem 1.1, it is known that $\mathrm{add}(\mathcal{M}) = \mathrm{add}(\mathcal{E})$, and moreover, a set $X \subseteq 2^\omega$ is meager-additive if and only if it is $\mathcal{E}$-additive [3]. The former equivalence is effectivized as the equivalence between lowness for weak 1-genericity and lowness for Kurtz randomness (see Stephan-Yu [72]). We can also effectivize the equivalence of meager-additivity and $\mathcal{E}$-additivity, which implies that all properties in items (4) and (5) of Theorem 1.1 are equivalent. However, it seems difficult to characterize Martin-Löf null-additivity by using traceability. See also Problem 7.2.

Fact 4.1 (see [44])**.** *Suppose $\mathbf{E} = \mathbf{E}_{\rm ce}$ and $(\mathcal{I}, \mathcal{J})$ is a good$_2$-pair (see Section 3.2). A set $A \subseteq \omega$ is $(\mathcal{I}, \mathcal{J}; \mathcal{C}_{\rm tt}, \mathbf{E})$-low$_{\rm test}$ if and only if A is $(\mathcal{S}, \tilde{\mathsf{Q}}; \mathcal{C}_{\rm tt}, \mathbf{E})$-traceable, where*

$$\mathcal{S} = \begin{cases} \mathcal{T}_{\rm semi} & \text{if } \mathcal{I} = \mathcal{N}_{\rm MLR}, \\ \mathcal{T}_{\rm tt} & \text{if } \mathcal{I} \in \{\mathcal{N}_{\rm SR}, \mathcal{E}_{\rm WR}\}, \end{cases} \qquad \tilde{\mathsf{Q}} = \begin{cases} \{\mathsf{Q}_{\rm cof}\} & \text{if } \mathcal{J} = \mathcal{N}_{\rm SR}, \\ \{\mathsf{Q}_{\rm inf}\} & \text{if } \mathcal{I} \neq \mathcal{J} = \mathcal{E}_{\rm WR}, \\ \{\mathsf{Q}_g : g \in \mathbf{E}\} & \text{if } \mathcal{I} = \mathcal{J} = \mathcal{E}_{\rm WR}. \end{cases}$$

For a locally KS set $\mathbf{E}$ and a pair $(\mathcal{I}, \mathcal{J})$, by Proposition 3.9, $(\mathcal{I}, \mathcal{J}; \mathcal{C}_{\rm tt}, \mathbf{E})$-lowness is equivalent to $(\mathcal{I}, \mathcal{J}; \mathcal{C}_{\rm tt}, \mathbf{E})$-smallness and hence, implies $(\mathcal{I}, \mathcal{J}; \mathbf{E})$-additivity. We also see by Lemma 3.12, $(\mathcal{S}, \tilde{\mathsf{Q}}; \mathcal{C}^\star_{\rm tt}, \mathbf{E})$-traceability is equivalent to $(\mathcal{S}, \tilde{\mathsf{Q}}; \mathcal{C}_{\rm tt}, \mathbf{E})$-traceability. We will show that

- $(\mathcal{I}, \mathcal{J}; \mathbf{E})$-additivity implies $(\mathcal{S}, \tilde{\mathsf{Q}}; \mathcal{C}^\star_{\rm tt}, \mathbf{E})$-traceability,
- $(\mathcal{S}, \tilde{\mathsf{Q}}; \mathcal{C}_{\rm tt}, \mathbf{E})$-traceability implies $(\mathcal{I}, \mathcal{J}; \mathcal{C}_{\rm tt}, \mathbf{E})$-low$_{\rm test}$-ness.

Combining this with previous observations, we will see the following:

Theorem 4.2. *Suppose that $\mathbf{E}$ is any locally KS set, and $(\mathcal{I}, \mathcal{J}, \mathcal{S}, \tilde{\mathsf{Q}})$ is any quadruple from Fact 4.1. Then, the following are equivalent for a set $A \subseteq \omega$:*

(i) *A is $(\mathcal{I}, \mathcal{J}; \mathcal{C}_{\rm tt}, \mathbf{E})$-low$_{\rm test}$.*
(ii) *A is $(\mathcal{J}, \mathcal{I})$-additive over $\mathbf{E}$.*
(iii) *A is $(\mathcal{S}, \tilde{\mathsf{Q}}; \mathcal{C}^\star_{\rm tt}, \mathbf{E})$-traceable.*

4.2. **Null-Additivity.** In this subsection, we show Theorem 4.2 when $\mathcal{I} = \mathcal{J} = \mathcal{N}$ endowed with certain representations. Franklin-Stephan [28] showed that a set $A \subseteq \omega$ is *tt*-low for Schnorr randomness (that is, $(\mathcal{N}_{\rm SR}, \mathcal{N}_{\rm SR}; \mathcal{C}_{\rm tt}, \mathbf{E}_{\rm ce})$-low) if and only if it is computably *tt*-traceable (that is, $(\mathcal{T}_{\rm tt}, \{\mathsf{Q}_{\rm cof}\}; \mathcal{C}_{\rm tt}, \mathbf{E}_{\rm ce})$-traceable). By Lemma 3.12, this is equivalent to computable $\star$-traceability (i.e., $(\mathcal{T}_{\rm tt}, \{\mathsf{Q}_{\rm cof}\}; \mathcal{C}^\star_{\rm tt}, \mathbf{E}_{\rm ce})$-traceable). The second author [53] generalized the above result in the context of uniform-low-for-Schnorr-randomness (ULSch) reducibility. We say that $A \subseteq \omega$ is *uniform-low-for-Schnorr-randomness reducible to $B \subseteq \omega$ over* $\mathbf{E}$ if a real is Schnorr random uniformly relative to A over $\mathbf{E}$ whenever it is Schnorr random uniformly relative to B over $\mathbf{E}$. We show the following implication by a straightforward modification of the standard method (see [24]).

Lemma 4.3. *Let A and B be subsets of ω. If A is $\mathcal{T}_{\rm tt}$-$\star$-traceable uniformly in B over $\mathbf{E}$, then A is uniform-low-for-Schnorr-randomness reducible to B over $\mathbf{E}$.*

Proof. Assume that A is $\mathcal{T}_r$-$\star$-traceable uniformly in B over $\mathbf{E}$. Note that every uniform Schnorr test $N : 2^\omega \to \mathcal{N}$ can be thought of as $Z \mapsto (N^{Z \upharpoonright u(n,m)}_{n,m})_{n,m\in\omega}$ such that

$$N(Z) = \bigcap_n \bigcup_m N^{Z \upharpoonright u(n,m)}_{n,m} \text{ and } \lambda(N^{Z \upharpoonright u(n,m)}_{n,m}) \in [2^{-n}(1 - 2^{-m-1}), 2^{-n}].$$

By uniform $\mathcal{T}_r$-$\star$-traceability, there is an $\mathbf{E}$-coded total continuous map $T : 2^\omega \to \mathcal{T}_r$, $Z \mapsto (T^Z_n)_{n\in\omega}$, such that $T^Z_{\langle n,m\rangle} \subseteq 2^{u(2n,2m)}$, $|T^Z_{\langle n,m\rangle}| \leq 2^n 2^m$ and $A \upharpoonright u(2n, 2m) \in T^B_{\langle n,m\rangle}$. Define $N_n(Z) = \bigcup_m \bigcup_{\sigma\in T^Z_{\langle n,m\rangle}} N^\sigma_{2n,2m}$.

Then,

$$\lambda(N_n(Z)) \leq \lambda\left(\bigcup_{\sigma\in T^Z_{\langle n,0\rangle}} N^\sigma_{2n,0}\right) + \sum_{m=1}^{\infty} \lambda\left(\bigcup_{\sigma\in T^Z_{\langle n,m\rangle}} N^\sigma_{2n,2m} \setminus N^\sigma_{2n,2m-2}\right)$$
$$\leq 2^n 2^{-2n} + \sum_{m=1}^{\infty}(2^n 2^m)(2^{-2n}2^{-2m}) = 2^{-n} + 2^{-n}\sum_{m=1}^{\infty} 2^{-m} = 2^{-n+1}.$$

Hence, $Z \mapsto \bigcap_n N_n(Z)$ is a uniform Schnorr (Martin-Löf, resp.) test over $\mathbf{E}$ if $T \in \mathcal{T}_{\rm tt}$ ($T \in \mathcal{T}_{\rm semi}$, resp.). Therefore, $N(A) \subseteq \bigcap_n N_n(B)$ is Schnorr null uniformly relative to B over $\mathbf{E}$. □

Shelah (see Bartoszyński-Judah [2, 3]) showed that a set $X \subseteq 2^\omega$ is $\star$-traceable (i.e., $(\mathcal{T}, \{\mathsf{Q}_{\rm cof}\}; \mathcal{C}^\star_{\rm tt}, \mathbf{V})$-traceable) if and only if it is null-additive (i.e., transitive $(\mathcal{N},\mathcal{N})$-additive over $\mathbf{V}$). The notion of $(\mathcal{I},\mathcal{J})$-additivity over $\mathbf{E}$ is introduced in Definition 2.36. Now we say that a set $V \subseteq 2^\omega$ is $(\mathcal{I},\mathcal{J};\mathcal{F},\mathbf{E})$-*additive relative to* B if

$$(\forall N \in \mathcal{I}^{\mathbf{E}})(\exists J \in \mathcal{F}^{\mathbf{E}}({}_{\subset}2^\omega, \mathcal{J}))\ V + N \subseteq J(B).$$

We say that V is $\mathbf{E}$-*Schnorr null-additive* $\mathcal{F}$-*uniformly in* B if V is $(\mathcal{N}_{\rm SR},\mathcal{N}_{\rm SR};\mathcal{F},\mathbf{E})$-additive relative to B. We effectivize Bartoszyński-Judah's proof of Shelah's theorem to show the following result (see [3]).

Lemma 4.4. *Let* $\mathbf{E}$ *be a locally KS set. Suppose that* $V \subseteq 2^\omega$ *is* $\mathbf{E}$*-Schnorr null-additive uniformly in* B*. Then,* $V \subseteq 2^\omega$ *is* $\mathcal{T}_{\rm tt}$*-*$\star$*-traceable over* $\mathbf{E}$ *uniformly in* B*, i.e.,* $(\mathcal{T}_{\rm tt}; \mathcal{C}^\star_{\rm tt}, \mathcal{C}_{\rm tt}, \mathbf{M})$*-traceable relative to* B*.*

Proof. Let $(\mathbf{E}_a)_{a\in\Lambda}$ is a KS-base of $\mathbf{E}$. We first assume that we have already constructed a $\mathbf{E}_a$-Schnorr null set $N = \bigcap_m \bigcup_n N_{m,n}$, where $\{N_{m,n}\}_{m,n}$ is an $\mathbf{E}_a$-coded sequence of clopen sets with measures are $\mathbf{E}_a$-coded as a sequence of reals, and $N_{m,n} \subseteq N_{m,n+1}$. By our assumption, we have a uniform $\mathbf{E}_b$-Schnorr test $J : 2^\omega \to \mathcal{N}_{\rm SR}$ mapping $Z \mapsto \bigcap_n J^Z_n$ such that $V + N \subseteq \bigcap_n J^B_n$.

Claim. Without loss of generality, if $W^Z = J^Z_t$ for a sufficiently large t, we may assume that $\{\sigma : [\sigma] \subseteq W^Z\}$ is $\mathbf{E}_b$-decidable uniformly in Z, $\lambda(W^Z) < 1/4$, and $\lambda(W^Z|\sigma) < 1$ whenever $[\sigma] \not\subseteq W^Z$.

By putting $W^Z = J^Z_t$ for a sufficiently large t, we may assume that $\lambda(W^Z) < 1/8$ for every $Z \subseteq \omega$. Then, the total continuous map $W : Z \mapsto W^Z$ can be identified with $W = \{(\sigma,\tau) : \sigma \prec Z,\ [\tau] \subseteq W^Z\}$. By compactness, for any n, there is a finite subset C_n of W such that $\lambda(W^Z \setminus C^Z_n) < 2^{-n}$ for all Z. By uniformization, one can find such an $\mathbf{E}_b$-coded sequence $(C_n)_{n\in\omega}$. Let $\{\sigma_k\}_{k\in\omega}$ be an effective enumeration of all binary strings. For every k, compute whether $\lambda(C^\tau_{k+5}|\sigma_k) \geq 1 - 2^{-(k+5)}$. If so, enumerate σ_k into U^τ. Since $\lambda(W^Z|\sigma_k) \geq 1 - 2^{-(k+4)}$,

$$\lambda(U^Z) \leq \lambda(W^Z) + \sum_k 2^{-|\sigma_k|}2^{-(k+4)} < 1/4.$$

Clearly, $U^Z \subseteq W^Z$ and this construction is uniformly $\mathbf{E}_b$-coded. Thus, the claim is verified by replacing W^Z with U^Z.

For every σ with $[\sigma] \not\subseteq W^Z$, consider the following:

$$V^Z_{\sigma,m}[n] = \{x \in 2^\omega : (x + N_{m,n}) \cap [\sigma] \subseteq W^Z \upharpoonright h(m,n)\},$$

where $h(m,n)$ is the least level such that the clopen set $N_{m,n}$ is determined, and $W^Z \upharpoonright h = \bigcup\{[\tau] : |\tau| \leq h \text{ and } [\tau] \subseteq W^Z\}$. Clearly, $h(m,n)$ is computable from the canonical index of $N_{m,n}$ uniformly in $m,n \in \omega$, and hence h is $\mathbf{E}_a$-coded. Now, each $V^Z_{\sigma,m}[n]$ is a clopen set determined at level $h(m,n)$. Therefore, $(V^Z_{\sigma,m}[n])_{\sigma,m,n}$ can be thought of as a sequence of (a canonical index of) a finite set of finite strings of length $h(m,n)$. Let $\mathbf{E}_c$ be an amalgamation of $\mathbf{E}_a$ and $\mathbf{E}_b$. Then, the $\mathbf{E}_b$-decidability of W implies that $Z \mapsto (V^Z_{\sigma,m}[n])_{\sigma,m,n}$ is an $\mathbf{E}_c$-coded total continuous map.

Claim. $V \subseteq \bigcup_{\sigma,m} \bigcap_n V^B_{\sigma,m}[n]$.

Indeed, if a G_δ set $N = \bigcap_n N_n$ is included in an open set W, there is $\sigma \notin W$ and m such that $N_m \cap [\sigma] \subseteq W$. Otherwise, put $\sigma_0 = \emptyset$, and for every m and $\sigma_m \notin W$, choose σ_{m+1} such that $\sigma_m \subseteq \sigma_{m+1} \in N_m \setminus W$ under the induction hypothesis $N_m \cap [\sigma_m] \not\subseteq W$. Then $\lim_m \sigma_m \in \bigcap_m N_m \setminus W$ contradicts our assumption $N \subseteq W$. As a consequence, $A \in V$ implies $A \in \bigcap_n V^B_{\sigma,m}[n]$ for some σ, m, since $\{A\} + N$ is G_δ and W^B is open.

Claim. Given $h(m) \in \omega$, one can effectively find a clopen set E_m of measure 2^{-m} such that for any distinct $\sigma, \tau \in 2^{h(m)}$, $E_m + \sigma$ and $E_m + \tau$ are measure independent.

Let $E_m \subseteq 2^{[h(m),h(m+1))}$ be the set of all strings of the form $\rho\sigma_k\tau$, where $|\rho| = h(m)$, σ_k is the k-th string of length $h(m)$, and $\tau[I_k] = 0^m$ for the k-th interval I_k of length m in $[h(m), h(m+1))$. Then the measure of E_m is clearly 2^{-m}.

Now we define a $\mathbf{E}$-Schnorr test. Let $h \in \omega^\omega$ be an arbitrary $\mathbf{E}$-coded (say, $\mathbf{E}_a$-coded) and then $(E_m)_{m\in\omega}$ is an $\mathbf{E}_a$-coded sequence of clopen sets as above. Define $N_n = \bigcup_{m>n} E_m$, and then it is an open set of measure 2^{-m}, and hence, $N = \bigcap_n N_n$ is $\mathbf{E}_a$-Schnorr null. Put $N_{m,n} = \bigcup_{m\leq k\leq n} E_k$. Then $V^Z_{\sigma,m}[n]$ is constructed from this $\mathbf{E}_a$-Schnorr null set as above.

Claim. If $\lambda(W^Z|\sigma) < 1/2$ and $|\sigma| \leq 2^{h(n)}$, then $V^Z_{\sigma,m}[n]\restriction h(n)$ has at most 2^n elements.

Now put $W^{Z+}_\sigma = V^Z_{\sigma,m}[n] + N_{m,n}$. Then, $W^{Z+}_\sigma \cap [\sigma] \subseteq W^Z\restriction h(n+1) \subseteq W^Z$. For every $\rho \in V^Z_{\sigma,m}[n]\restriction h(n)$, the measure of $\rho + N_{m,n}$ is at least 2^{-n}, and by their probabilistically independence, we have

$$\frac{1}{2} > \lambda(W^Z|\sigma) \geq \lambda(W^{Z+}_\sigma|\sigma) = 1 - \prod_{\tau\in V^Z_{\sigma,m}[n]} \lambda(\tau + (2^\omega \setminus N_m)) \geq 1 - \left(1 - \frac{1}{2^n}\right)^{|V^Z_{\sigma,m}[n]\restriction h(n)|}.$$

Meanwhile, for $k = 2^n$ and $m \geq k-1$, we have

$$\left(1 - \frac{1}{k}\right)^m \leq \left(1 - \frac{1}{k}\right)^{k-1} = \left(\left(1 + \frac{1}{k-1}\right)^{k-1}\right)^{-1} \leq \frac{1}{2}.$$

Hence, $|V^Z_{\sigma,m}[n]\restriction h(n)^+| < 2^n - 1$.

Consider the following trace:

$$T^Z_n = \bigcup\{V^Z_{\sigma,m}[n]\restriction h(n) : \langle\sigma, m\rangle \leq n, |V^Z_{\sigma,m}[n]\restriction h(n)| \leq 2^n\}$$

Clearly $|T^Z_n| \leq n2^n$, and if $A \in V$ then $A\restriction h(n) \in T^B_n$ for almost all n since every σ with $[\sigma] \not\subseteq W^Z$ can be extended to σ^+ with $\lambda(W^Z|\sigma^+) < 1/2$. Moreover, $Z \mapsto T^Z_n \in \mathcal{T}_{\rm tt}$ is an $\mathbf{E}_c$-coded total continuous map, since $Z \mapsto V^Z_{\sigma,m}[n]$ is an $\mathbf{E}_c$-coded total continuous map as discussed above. □

To show Theorem 4.2 for $\mathcal{I} = \mathcal{N}_{\rm MLR}$ and $\mathcal{N}_{\rm SR}$, we just modify the proof of Lemma 4.4. We say that V is (Schnorr,Martin-Löf)-null-additive over $\mathbf{E}$ if it is $(\mathcal{N}_{\rm SR}, \mathcal{N}_{\rm MLR})$-additive over $\mathbf{E}$.

The set $[\omega]^{<\omega}$ of all finite subsets of ω is ω-represented (via the canonical index); therefore, we can identify $\mathcal{P}[\omega]^{<\omega}$ with $\mathcal{O}\omega$. A set $\mathcal{A} \subseteq \mathcal{P}[\omega]^{<\omega}$ is *hereditary* if $A \subseteq B \in \mathcal{A}$ implies $A \in \mathcal{A}$. We say that a pointclass Γ satisfies the *shuttering property* if the following condition holds: Given $V \subseteq \mathcal{X} \times \omega$ in Γ and a hereditary set $\mathcal{A} \subseteq \mathcal{P}[\omega]^{<\omega}$ where $\mathcal{A} \in \Gamma$, there exists a Γ set $V^* \subseteq V$ such that $V^*_x \in \mathcal{A}$, and that $V_x \in \mathcal{A}$ implies $V^*_x = V_x$. Here, $V_x = \{(x, n) : (x, n) \in V\}$.

Lemma 4.5. *Any Kleene-or-Spector pointclass satisfies the shuttering property.*

Proof. If Γ is a Kleene pointclass, it is clear. Suppose that Γ is a Spector pointclass. Let φ be a Γ-norm on V (see Moschovakis [55]). Consider V^* defined as the set of all $(x, n) \in V$ satisfying that for any finite set $D \subseteq \omega$ with $|D| \notin \mathcal{A}$ there is $m \in D$ such that either $(x, m) \equiv^\varphi_\Gamma (x, n)$ and $n < m$ or $(x, m) \not\leq^\varphi_\Gamma (x, n)$. Clearly, V^* is in $\forall^\omega\Gamma$, and satisfies the desired condition. □

Lemma 4.6. *Let* $\mathbf{E}$ *be a locally KS set. If* V *is (Schnorr, Martin-Löf)-null-additive over* $\mathbf{E}$, *then* $V \subseteq 2^\omega$ *is* $\mathcal{T}_{\rm semi}$-$\star$*-traceable over* $\mathbf{E}$.

Proof. In the proof of Lemma 4.4, we can satisfy all claims except for the $\mathbf{E}_b$-decidability in the first claim. For instance, to satisfy $\lambda(W^Z|\sigma) < 1$ whenever $[\sigma] \not\subseteq W^Z$, we enumerate all σ_k such that there is a clopen set $C \subseteq W$ such that $\lambda(C|\sigma_k) \geq 1 - 2^{-k-4}$. Then, by Lemma 4.5, replace $V_{\sigma,m}[n]\restriction h(n)$ with $V^*_{\sigma,m}[n]\restriction h(n)$, the set of the first 2^n elements (w.r.t. a Γ-norm) satisfying $(x + N_{n,m}) \cap [\sigma] \subseteq W\restriction h(m, n)$. It is not hard to verify that the same strategy works. □

Lemmata 4.3, 4.4 and 4.6 clearly imply Theorem 1.1 (1) and (2). Furthermore, generally, null-additivity is equivalent to uniform lowness for Schnorr randomness at the levels of arithmetic, hyperarithmetic (Δ^1_1), Δ^1_2, infinite time register machine computability, infinite time Turing machine computability, etc. For instance, ITTM-null additivity is equivalent to lowness for ITTM-Schnorr randomness w.r.t. uniform relativization.

Remark 4.7. It is commonly known that the notion of traceability/slalom has an ℓ_1-characterization in the both areas of algorithmic randomness theory and set theory [3]. Here, ℓ_1 consists of *summable* sequences of reals. Bienvenu-Miller [6] gave the ℓ_1-characterizations of Low(MLR) (which is equivalent to Low*(MLR)), Low(SR) and Low(MLR, SR). We also have the ℓ_1-characterizations of Low*(SR) and Low*(MLR, SR) (see [44, 53]). It is not hard to see that the same characterizations hold over any principal KS set.

4.3. **$\mathcal{E}$-Additivity.** Next, we show Theorem 4.2 when $\mathcal{I} = \mathcal{J} = \mathcal{E}$ endowed with the standard F_σ representation. Kihara-Miyabe [43] showed that a set is uniformly low for Kurtz randomness (i.e., $(\mathcal{E}_{\rm WR}, \mathcal{E}_{\rm WR}; \mathcal{C}_{\rm tt}, \mathbf{E}_{\rm ce})$-low) if and only if it is computably c.o. tt-traceable (i.e., it is $(\mathcal{T}_{\rm tt}, \{\mathsf{Q}_g : g \in (\omega^\omega)^{\mathbf{E}_{\rm ce}}\}; \mathcal{C}_{\rm tt}, \mathbf{E}_{\rm ce})$-traceable). We say that $A \subseteq \omega$ is *uniform-low-for-Kurtz-randomness reducible to* $B \subseteq \omega$ *over* $\mathbf{E}$ if a real is Kurtz random uniformly relative to A over $\mathbf{E}$ whenever it is Kurtz random uniformly relative to B over $\mathbf{E}$. One can easily show the following implication by a straightforward modification of the standard method (see [24, 43]).

Lemma 4.8. *Let A and B be subsets of ω. If A is $\mathbf{E}$-often $\mathcal{T}_{\rm tt}$-$\star$-traceable uniformly in B over $\mathbf{E}$ (i.e., $(\mathcal{T}_{\rm tt}, \{\mathsf{Q}_g : g \in (\omega^\omega)^{\mathbf{E}}\}; \mathcal{C}_{\rm tt}, \mathcal{C}_{\rm tt}, \mathbf{E}_{\rm ce})$-traceable relative to B), then A is uniform-low-for-Kurtz-randomness reducible to B over $\mathbf{E}$.* □

It is natural to ask whether there is also an additive characterization of uniform lowness for Kurtz randomness. We affirmatively answer this question by modifying Pawlikowski's additive characterization of strong measure zero [59]. We say that V is $\mathbf{E}$-*Kurtz null-additive $\mathcal{F}$-uniformly in B* if V is $(\mathcal{E}_{\rm WR}, \mathcal{E}_{\rm WR}; \mathcal{F}, \mathbf{E})$-additive relative to B (see Section 4.2 for the definition).

Lemma 4.9. *Suppose that V is $\mathbf{E}$-Kurtz null-additive uniformly relative to B. Then, V is $\mathbf{E}$-often $\star$-traceable uniformly relative to B over $\mathbf{E}$.*

Proof. Let $(\mathbf{E}_a)_{a\in\Lambda}$ is a KS-base of $\mathbf{E}$. We first define a special $\mathbf{E}_a$-Kurtz test D which has some probabilistic independence property.

Construction of an $\mathbf{E}_a$-Kurtz test D. Given an $\mathbf{E}_a$-coded order $g \in \omega^\omega$, inductively define two $\mathbf{E}_a$-coded orders h and h^+ by $h(0) = 0$, $h^+(n) = h(n) + g(n)$, and $h(n+1) = h^+(n) + 2^{g(n)}$. Then let $D_k \subseteq 2^{h(k+1)}$ be the set of all strings of the form $\tau^\frown\sigma_i{}^\frown\rho$ such that $\tau \in 2^{h(k)}$, $\rho(i) = 0$, and σ_i is the i-th string in $2^{[h(k),h(k)^+)}$. Then $D = \bigcap_n [\![D_n]\!]$ is an $\mathbf{E}_a$-Kurtz test, since the λ-measure of $\bigcap_{m<n}[D_m]$ and $[D_n]$ are 2^{-n} and 2^{-1} respectively.

By our assumption, there is a uniform $\mathbf{E}_b$-Kurtz test E such that $V + D = \bigcup_{A\in V}(A + D)$ is covered by $E(B) = \bigcap_n E^B_n$, where $\lambda(E^Z_n) \le 2^{-n}$ for every Z. Given k, by compactness, there is $e(k)$ such that $\lambda(E^Z_{e(k)}) < 2^{-3} \cdot 2^{-h(k)}$ for every $Z \subseteq \mathbb{N}$. By uniformization, such an $\mathbf{E}_b$-coded sequence $(e(k))_{k\in\omega}$ exists. Put $d(0) = e(0)$, $d(k+1) = e(k+1)$ if $E^Z_{d(k)} \subseteq 2^{\le h(k)}$ for all $Z \subseteq \mathbb{N}$, and put $d(k+1) = d(k)$ otherwise. Note that d is unbounded, by uniformity and compactness.

Now given $\tau \in 2^{h(k)}$, define $E^Z_\tau[k] \subseteq 2^{[h(k),h(k+1))}$ as follows:

$$E^Z_\tau[k] = \left\{\sigma \in 2^{[h(k),h(k+1))} : (1 - 2^{-(k+1)}) \cdot \lambda(E^Z_{d(k)}|\tau\sigma) > \lambda(E^Z_{d(k)}|\tau)\right\}.$$

Then, define $V^B_\tau[k] \subseteq 2^{[h(k),h(k)^+)}$ as follows:

$$V^Z_\tau[k] = \left\{\sigma \in 2^{[h(k),h(k)^+)} : (\exists\sigma^+ \in 2^{[h(k),h(k+1))})\ \sigma \preceq \sigma^+ \text{ and } \sigma^+ + D_k \subseteq E^Z_\tau[k]\right\}.$$

Finally, put $V^Z[k] = \bigcup_{\tau\in 2^{h(k)}} V^Z_\tau[k]$. If $\mathbf{E}_c$ is an amalgamation of $\mathbf{E}_a$ and $\mathbf{E}_b$, then clearly, $Z \mapsto V^Z$ is an $\mathbf{E}_c$-coded total continuous map.

Claim. $\#V^Z[k] \le (k+1) \cdot 2^{h(k)}$ for every $Z \subseteq \mathbb{N}$.

By definition, $V^Z_\tau[k] + D_k \subseteq E^Z_\tau[k]$. By probabilistic independence of D_k,

$$\lambda(V^Z_\tau[k] + D_k) = 1 - 2^{-|V^Z_\tau[k]|},$$

while by Kolmogorov's inequality,

$$\lambda(E^Z_\tau[k]) < 1 - 2^{-(k+1)}.$$

This implies $\#V^Z_\tau[k] \leq k+1$. Hence, the desired value is computed by multiplying the above number by the number of strings $\tau \in 2^{h(k)}$.

Now, let $\{l(m)\}_{m\in\omega}$ be the list of all k's such that $d(k) \neq d(k-1)$. Clearly, $\{l(m)\}_{m\in\omega}$ is an $\mathbf{E}_b$-coded (hence, $\mathbf{E}_c$-coded) sequence.

Claim. $V \subseteq \bigcap_m \bigcup_{k=l(m)}^{l(m+1)-1} V^B[k]$.

Suppose for the sake of contradiction that there is $A \in V$ such that $A \notin \bigcup_{k=l(m)}^{l(m+1)-1} V^B[k]$ for some m. Put $s = l(m)$ and $t = l(m+1)-1$. By our definition of d, we have

$$E^B_{d(s)} \subseteq 2^{\leq h(t)}, \text{ and } \lambda(E^B_{d(s)}|A+\tau) < 2^{-3} \text{ for every } \tau \in 2^{h(s)}.$$

Since $V + D \subseteq E^B_{d(s)} \subseteq 2^{\leq h(t)}$, we must have

$$\lambda(E^B_{d(s)}|A+\tau) = 1 \text{ for every } \tau \in 2^{h(t)}.$$

However, this is impossible because $A \notin \bigcup_{k=s}^{t} V^B[k]$ implies that one can construct $Z \in D$ fulfilling

$$(1-2^{-(k+1)}) \cdot \lambda(E^B_{d(s)}|A + Z\restriction h(k+1)) \leq \lambda(E^B_{d(s)}|A + Z\restriction h(k))$$

for every $k \in [s,t)$, and this implies that

$$\prod_{i=k_0}^{k-1}(1-2^{-(i+1)}) \cdot \lambda(E^B_{d(s)}|A + Z\restriction h(t)) \leq \lambda(E^B_{d(s)}|A + Z\restriction h(s)) < 2^{-3}.$$

Thus, we have $\lambda(E^B_{d(s)}|A + Z\restriction h(t)) < 1$, and this contradicts our assumption $A \in V$. In particular, V is $\mathbf{E}$-often piecewise traceable uniformly in B over $\mathbf{E}$. □

Lemmata 4.8 and 4.9 clearly imply Theorem 1.1 (5). As a consequence, Kurtz null-additivity (Kurtz random-preserving) is equivalent to uniform lowness for Kurtz randomness at the levels of computability, arithmetic, hyperarithmetic (Δ^1_1), Δ^1_2, infinite time register machine computability, infinite time Turing machine computability, etc. For instance, A is ITRM-Kurtz-randomness preserving iff A is low for ITRM-Kurtz randomness w.r.t. uniform relativization.

4.4. **Meager-Additivity.** In this subsection, we show Theorem 4.2 when $\mathcal{I} = \mathcal{J} = \mathcal{M}$ endowed with the standard F_σ representation. Our proof is the straightforward effectivization of the standard argument. To show the theorem, we effectivize the well-known characterization of meager sets.

Definition 4.10 (see Bartozyński-Judah [3, Theorem 2.2.4])**.** For an order u, by $M^x_{u,s}$ be the set of all reals which are not piecewise u-traced by a real $x \in 2^\omega$ at all levels above s, that is,

$$M^x_{u,s} = \{y \in 2^\omega \ (\forall n \geq s)\ x\restriction[u(n), u(n+1)) \neq y\restriction[u(n), u(n+1))\}.$$

Put $M^x_u = \bigcup_s M^x_{u,s}$. Then M^x_u is the set of all reals which are not i.o. piecewise u-traced by x (see Section 3.3).

Lemma 4.11. *A set $X \subseteq 2^\omega$ is meager over* $\mathbf{E}$ *(i.e., $X \in \mathcal{M}^{\mathbf{E}}_{\rm W1G}$) if and only if X is included in M^x_u for some* $\mathbf{E}$*-coded real $x \in 2^\omega$ and* $\mathbf{E}$*-coded order $u \in \omega^\omega$.*

Proof. ($\Leftarrow$): Clearly M^x_u is the union of the $\mathbf{E}$-coded sequence $\{M^x_{u,s}\}_{s\in\omega}$ of nowhere dense closed sets.

($\Rightarrow$): If X is meager over $\mathbf{E}$, then it is included in the union of a uniform sequence $\{F_n\}_{n\in\omega}$ of $\mathbf{E}$-semicoded closed sets. For any n and $\sigma \in 2^n$, there is ρ_σ such that $[\sigma^\frown\rho_\sigma] \cap \bigcup_{k\leq n} F_k = \emptyset$ by nowhere density. Let ρ_n be the concatenation of all ρ_σ for $\sigma \in 2^n$. Thus, for any n there is ρ_n such that $[\sigma^\frown\rho_\sigma] \cap \bigcup_{k\leq n} F_k = \emptyset$ for all $\sigma \in 2^n$. By uniformization, such a sequence $(\rho_n)_{n\in\omega}$ is $\mathbf{E}$-coded. Then, define x_0 to be an empty string, and $x_{n+1} = x_{|\alpha_n|}$. It is not hard to see that $X \subseteq M^x_u$, where $x = \lim_n x_n$ and $u(n) = |x_n|$. □

We say that V is *meager-additive* over $\mathbf{E}$ if it is $(\mathcal{M}_{\rm W1G}, \mathcal{M}_{\rm W1G})$-additive over $\mathbf{E}$.

Lemma 4.12. *A set $X \subseteq 2^\omega$ is meager-additive over* $\mathbf{E}$ *if and only if X is* $\mathbf{E}$*-often piecewise traceable by an* $\mathbf{E}$*-coded real.*

Proof. By Lemma 4.11, every $\mathbf{E}$-semicoded meager set is contained in a set of the form M_u^C for some order u and $\mathbf{E}$-coded real C. Note that $X + M_u^C = \bigcup_{A\in X} M_u^{A\triangle C}$. Moreover, $M_u^{A\triangle C} \subseteq M_v^B$ if and only if $M_u^A \subseteq M_v^{B\triangle C}$. Therefore, X is meager-additive over $\mathbf{E}$ if and only if for every $\mathbf{E}$-coded order u, there are an $\mathbf{E}$-coded order v and an $\mathbf{E}$-coded real B such that $\bigcup_{A\in X} M_u^A \subseteq M_v^B$, that is, for every $A \in X$, $M_u^A \subseteq M_v^B$.

The last condition $M_u^A \subseteq M_v^B$ is equivalent to the condition that almost every interval $[v(n), v(n+1))$ contains an interval $[u(k), u(k+1))$ in which A agrees with B. Otherwise, we can construct B^* which agrees with B in each such interval $[v(n), v(n+1))$ and is contained in M_u^A. This implies $M_u^A \not\subseteq M_v^B$. Thus, by using v, we may effectively find an $\mathbf{E}$-coded order g such that B piecewise u-traces A g-often. □

We here refer to $\mathcal{M}_{\mathrm{W1G}}$-genericity as Cohen quasigenericity.

Lemma 4.13. *Let* $\mathbf{E}$ *be a locally KS set. If $A \subseteq \omega$ is* $\mathbf{E}$*-often traceable over* $\mathbf{E}$*, then A is low for Cohen quasigenericity w.r.t. uniform relativization over* $\mathbf{E}$*.*

Proof. Suppose that an $\mathbf{E}$-coded total continuous map $Z \mapsto M^Z \in \mathcal{M}$ is given. We simply assume that M^Z is nowhere dense closed set. By compactness, for all $n \in \omega$, there are $l_n, s_n \in \omega$ and a finite function $\rho_n : 2^{s_n} \to 2^{l_n}$ such that for any $Z \in 2^\omega$ and $\sigma \in 2^n$, $M^Z \cap [\sigma^\frown\rho_n(Z\upharpoonright s_n)] = \emptyset$. By uniformization, we have such an $\mathbf{E}$-coded sequence $(l_n, s_n, \rho_n)_{n\in\omega}$. Then, note that

$$M^Z \subseteq \{Z \in 2^\omega : (\forall n)\ Z\upharpoonright[n, n+l_n) \neq \rho_n(Z\upharpoonright n)\}.$$

Now we define $k(0) = 0$, $k(n+1) = k(n) + l_{k(n)}$, and $t(n) = \sum_{i=0}^n i$. By $\mathbf{E}$-often traceability, there are an $\mathbf{E}$-coded slalom $(T_n)_{n\in\omega}$ and $\mathbf{E}$-coded order $u \in \omega^\omega$ such that for all $k \in \omega$, $A\upharpoonright s_{k(t(n+1))} \in T_n$ for some $n \in [u(k), u(k+1))$, where we may assume that $|T_n| \leq n$. Then, we define

$$\begin{aligned} E^n_\tau = \{Z \in 2^\omega : (\forall j \leq n)\ & \\ & Z\upharpoonright[k(t(n)+j), k(t(n)+j+1)) \neq \rho_{k(t(n)+j)}(\tau\upharpoonright s_{k(t(n)+j)})\}, \end{aligned}$$

and set $E = \bigcap_k \bigcup_{n\in[u(k),u(k+1))} \bigcup_{\tau\in T_n} E^n_\tau$. It is not hard to verify that $M^A \subseteq E$. We can also show nowhere density of E. □

Lemmata 4.12 and 4.13 clearly imply Theorem 1.1 (6). As a consequence, meager-additivity (Cohen quasigenericity preserving) is equivalent to uniform lowness for Cohen quasigenericity at the levels of computability, arithmetic, hyperarithmetic (Δ^1_1), Δ^1_2, infinite time register machine computability, infinite time Turing machine computability, etc. For instance, Δ^1_1-meager-additivity is equivalent to lowness for Δ^1_1-Cohen-quasigenericity w.r.t. uniform relativization.

4.5. **Strong Measure Zero.** In this subsection, we show Theorem 4.2 when $\mathcal{I} = \mathcal{N}$ and $\mathcal{J} = \mathcal{E}$ endowed with suitable representations. Borel's original definition of *strong measure zero* is clearly equivalent to i.o. $\star$-traceability. Pawlikowski [59] showed that X is of strong measure zero (i.o. $\star$-traceable) if and only if X is $(\mathcal{E}, \mathcal{N})$-additive, that is, for every closed null set $E \in \mathcal{E}$, $X + E$ is null. The notion of strong measure zero is effectivized by Higuchi-Kihara [32] to characterize Binns' notion of effective perfect set property. Kihara-Miyabe [44] gave a dimension-theoretic characterization of $\mathrm{Low}^\star(\mathrm{SR}, \mathrm{WR})$ by using Pawlikowski's characterization of strong measure zero, and indeed, their proof essentially gave the additive characterization of strong measure zero.

Lemma 4.14 (see Kihara-Miyabe [44])**.** *Let* $\mathbf{E}$ *be a locally KS set. Then, (Kurtz, Schnorr)-null additivity over* $\mathbf{E}$ *implies i.o. $\mathcal{T}_{\mathrm{tt}}$-$\star$-traceable over* $\mathbf{E}$*, and i.o. $\mathcal{T}_{\mathrm{tt}}$-$\star$-traceable over* $\mathbf{E}$ *implies* low$_{\mathrm{test}}$*-ness for (Schnorr, Kurtz)-randomness over* $\mathbf{E}$ *w.r.t. uniform relativization.* □

By the same argument, we also have the Martin-Löf random version of the above result.

Lemma 4.15 (see Kihara-Miyabe [44])**.** *Let* $\mathbf{E}$ *be a locally KS set. Then, (Kurtz, Martin-Löf)-null additivity over* $\mathbf{E}$ *implies i.o. $\mathcal{T}_{\mathrm{semi}}$-$\star$-traceable over* $\mathbf{E}$*, and i.o. $\mathcal{T}_{\mathrm{semi}}$-$\star$-traceable over* $\mathbf{E}$ *implies* low$_{\mathrm{test}}$*-ness for (Martin-Löf, Kurtz)-randomness over* $\mathbf{E}$ *w.r.t. uniform relativization.* □

Lemmata 4.14 and 4.15 clearly imply Theorem 1.1 (3) and (4). As a consequence, $(\mathcal{E},\mathcal{N})$-additivity (strong measure zero) is equivalent to uniform lowness for $(\mathcal{N},\mathcal{E})$-tests at the levels of computability, arithmetic, hyperarithmetic (Δ^1_1), Δ^1_2, infinite time register machine computability, infinite time Turing machine computability, constructibility (in the sense of Gödel's L), etc. For instance, (Δ^1_2-Kurtz, Σ^1_2-Martin-Löf)-null additivity is equivalent to lowness for (Δ^1_2-Kurtz, Σ^1_2-Martin-Löf)-randomness w.r.t. uniform relativization.

Remark 4.16. Note that the cardinal characteristic $\mathrm{cov}(\mathcal{M})$ in Cichoń's diagram is known to be characterized as $\mathrm{add}(\mathcal{E},\mathcal{N})$ (see [3]). For their uniform versions, the similar equivalence $\mathrm{cov}^\star(\mathcal{M}) = \mathrm{add}^\star(\mathcal{E},\mathcal{N})$ holds. In other words, $X \subseteq 2^\omega$ is strong measure zero if and only if for every meager set $F \subseteq 2^\omega$, $X + F \neq 2^\omega$ (see Bartozyński-Judah [3, Theorem 8.1.16]). One can effective this result in a straightforward manner.

5. Uniform Lowness for Randomness

5.1. **Levels of Uniformity.** In previous works in algorithmic randomness theory, two uniformity levels of lowness properties have been investigated, that is, Low (nonuniform) and $\mathrm{Low}^\star$ (uniform). Since computability is the lightface version of continuity, the former notion Low is associated with partial continuity, and the latter notion $\mathrm{Low}^\star$ is associated with (uniform) continuity on an effectively compact domain. Generally, our inner universe $\mathbf{E} = \mathbf{E}_\Gamma$ involves its own uniformity level $\mathcal{F}_\Gamma$, which is generally much less uniform than partial continuity, and hence, one can deal with higher levels of non-uniformity, such as Borel and Π^1_1-measurable. For instance, the natural uniform level for $\mathbf{E}_{\Pi^1_1}$ is partial $\mathbf{\Pi}^1_1$-measurability; however, Bienvenu-Greenberg-Monin [5] emphasized continuous-uniformity for Π^1_1-Martin-Löf randomness, and characterized it by higher K-triviality.

There are also known traceability characterization of $(\mathcal{I},\mathcal{J};\mathcal{F}_{\mathbf{\Pi}^1_1},\mathbf{E}_{\Pi^1_1})$-lowness (see [19, 46]) in randomness theory over $\mathbf{E}_{\Pi^1_1}$. It is natural to ask whether the similar characterizations hold for other levels of uniformity, e.g., continuous-uniformity, Borel-uniformity, etc. Generally, for a locally KS set $\mathbf{E}$, we say that a represented function space $\mathcal{G}$ is $\mathbf{E}$-*good′* if $\mathcal{F}$ is closed under taking composition, and the inclusion map $\mathcal{C}(\subseteq \omega^\omega,\omega^\omega) \hookrightarrow \mathcal{F}(\subseteq \omega^\omega,\omega^\omega)$ is $\mathbf{E}$-semicoded. In Section 5, we will show the following:

Theorem 5.1. *Suppose that $\mathbf{E}$ is a locally KS set, $\mathcal{F}$ is an $\mathbf{E}$-good′ function space, $\mathcal{I} = \mathcal{J} \in \{\mathcal{N}_{\mathrm{SR}}, \mathcal{E}_{\mathrm{WR}}, \mathcal{M}_{\mathrm{W1G}}\}$, and moreover, $(\mathcal{S},\tilde{\mathrm{Q}})$ fulfills the condition in Fact 4.1. Then, $(\mathcal{I},\mathcal{J};\mathcal{F},\mathbf{E})$-lowness is equivalent to $(\mathcal{S},\tilde{Q};\mathcal{F},\mathbf{E})$-traceability.*

5.2. **Total Representations.** We first see that the representations ρ_{SR}, ρ_{WR}, and ρ_{W1G} admit total ω^ω-representations. Here, we often identify an ω^ω-representation ρ with an $\mathcal{O}\omega$-representation $\rho \circ \rho_{\omega^\omega}$. A multi-representation $\rho :\subseteq \mathcal{O}\omega \to \mathcal{I}$ is *computably reducible* to a multi-representation $\rho' :\subseteq \mathcal{O}\omega \to \mathcal{I}$ if there is an enumeration operator $h : \mathrm{dom}(\rho) \to \mathrm{dom}(\rho')$ such that $\rho(p) \subseteq \rho'(h(p))$ for every $p \in \mathrm{dom}(\rho)$. Two multi-representations are *computably equivalent* if they are computably reducible to each other.

Lemma 5.2. *There are total ω^ω-representations which are computably equivalent to ρ_{SR}, ρ_{WR}, and ρ_{W1G}.*

Proof. Fix a effective sequence $\langle C^n_m\rangle_{m,n\in\omega}$, where $\{C^n_m\}_{m\in\omega}$ is the set of all clopen sets in 2^ω of measure 2^{-n}. Then, we define the total ω^ω-representations ρ'_{SR} and ρ'_{WR} as follows:

$$\rho'_{\mathrm{SR}}(p) = \bigcap_{n\in\omega}\bigcup_{k\in\omega}[\![C^{n+k+1}_{p(\langle n,k\rangle)}]\!],$$

$$\rho'_{\mathrm{WR}}(p) = \bigcap_{n\in\omega}[\![C^n_{p(n)}]\!].$$

Given $p \in \mathrm{dom}(\rho_{\mathrm{SR}})$, for any $n,k \in \omega$, there is $e \in \omega$ such that $D_e \subseteq p^{[n]}$ and $2^{-n-k} \leq \lambda(\bigcup_{i\in D_e}[\sigma_i]) \leq 2^{-n}$, where σ_i is the i-th binary string. Thus, ρ_{SR} is computably reducible to ρ'_{SR}. One can also easily see that ρ'_{SR} is computably reducible to ρ_{SR} by enumerating all indices of $C^{n+k+1}_{p\langle n,k\rangle}$ into the n-th section. By the same argument, it is easy to see that ρ'_{WR} is equivalent to ρ_{WR}. To give a total ω^ω-representation of ρ_{W1G}, we define $u : \omega^\omega \to \omega^\omega$ by $u(p)(0) = p(0)$ and $u(p)(n+1) = \max\{p(n+1), u(p)(n)+1\}$. Clearly, $u(p)$ is strictly increasing for every $p \in \omega^\omega$. Then, we define the total representation $\rho'_{\mathrm{W1G}} : 2^\omega \times \omega^\omega \to \mathcal{M}$ as follows:

$$\rho'_{\mathrm{W1G}}(\langle p,q\rangle) = M^p_{u(q)}.$$

Here, M^A_u is the meager set generated by a real A and a strictly increasing function u defined by Section 4.4. Lemma 4.11 shows the equivalence of ρ_{W1G} and ρ'_{W1G}. ☐

5.3. $\mathrm{SR}^{\mathcal{N}}$ **Sets.** Recław [61] studied some continuous and Borel versions of additivity. Bartoszyński and Judah [2] introduced the notion of $\mathrm{SR}^{\mathcal{M}}$ sets and $\mathrm{SR}^{\mathcal{N}}$ sets. Obviously, the notion $\mathrm{SR}^{\mathcal{M}}$ is related to Borel-uniform lowness for Cohen quasigenericity (or equivalent to $\mathrm{Add}^{\mathcal{B}}(\mathcal{M})$), and $\mathrm{SR}^{\mathcal{N}}$ is related to Borel-uniform lowness for randomness (or equivalent to $\mathrm{Add}^{\mathcal{B}}(\mathcal{N})$). Therefore, one can introduce the $\mathcal{F}$-uniform versions of $\mathrm{SR}^{\mathcal{M}}$ sets and $\mathrm{SR}^{\mathcal{N}}$ sets for an arbitrarily given uniformity constraint $\mathcal{F}$. Then, we can easily effectivize the argument by Bartoszyński-Judah [2], and show that the uniformity level of lowness is strongly related to the uniformity level of traceability. We first show that $(\mathcal{N}_{\mathrm{SR}}, \mathcal{N}_{\mathrm{SR}}, \mathcal{F}, \mathbf{E})$-lowness is equivalent to $(\mathcal{T}_{\mathrm{tt}}, \{\mathsf{Q}_{\mathrm{cof}}\}, \mathcal{F}, \mathbf{E})$-traceability.

Theorem 5.3. *A set $A \subseteq \mathbb{N}$ is low for $\mathbf{E}$-Schnorr randomness w.r.t. $\mathcal{F}$-uniform relativization if and only if every $g \leq^{\mathbf{E}}_{\mathcal{F}} A$ is $\mathcal{T}_{\mathrm{tt}}$-traceable over $\mathbf{E}$.*

Proof. For the "only if" direction, we effectivize the argument in Bartoszyński-Judah [2, Theorem 3.7]. Suppose that $A \subseteq \mathbb{N}$ is $\mathcal{F}$-uniformly low for randomness over $\mathbf{E}$. Fix $g \leq_{\mathcal{F}} A$ via $\Phi \in \mathcal{F}^{\mathbf{E}}$, and consider

$$H_g = \{x \in 2^\omega : (\exists^\infty n)\ x \upharpoonright [g(n), g(n)+n) = \vec{0}\}.$$

Clearly, $A \mapsto H_{\Phi(A)} = H_g$ is an $\mathcal{F}$-uniform $\mathcal{N}_{\mathrm{SR}}$-test over $\mathbf{E}$. Therefore, $\mathcal{F}$-uniform lowness of A, there is a $\mathbf{E}$-semicoded open set G such that $\lambda(G) < 1/2$ and $H_g \subseteq G$. Now, as in Miller [50, Lemma 5], given an open set $G \subseteq 2^\omega$ put $G_n = \bigcup\{[\sigma] : \sigma \in 2^{<\omega},\ [\sigma] \subseteq G\}$, and choose a rapidly increasing sequence $(\varepsilon_n)_{n\in\omega}$. Let f_G be any function satisfying $\lambda(G \setminus G_{f_G(n)}) < \varepsilon_n$ for all $n \in \omega$. Then we can effectivize a theorem by Miller [50, Lemma 5] as the statement that if G is an $\mathbf{E}$-semicoded open set of measure less than $1/2$ such that $H_g \subseteq G$, then $g(n) < f_G(n)$ for almost all $n \in \omega$. Note that by uniformization, f_G can be assumed to be $\mathbf{E}$-coded whenever $\lambda(G)$ is $\mathbf{E}$-coded. By modifying f_G, we have an $\mathbf{E}$-coded function $b \in \omega$ such that $g(n) < b(n)$ for all $n \in \omega$. Let $z_g \in 2^\omega$ be a real satisfying

$$z_g \upharpoonright [b^*(n-1), b^*(n)) = 0^{g(n)} 1 0^{b(n)-g(n)-1},$$

where $b^*(0) = 0$ and $b^*(n) = \sum_{m<n} b(m)$. The map $g \mapsto z_g$ is effectively extended to an $\mathbf{E}$-coded total continuous function. In particular, $z_g \leq^{\mathbf{E}}_{\mathcal{F}} A$ since $\mathcal{C}_{\mathrm{tt}}$ is effectively embedded into $\mathcal{F}$. Now it is easy to see that z_g is $\mathcal{F}$-uniformly low for randomness over $\mathbf{E}$ since this property is $\leq^{\mathbf{E}}_{\mathcal{F}}$-downward closed. By Lemma 4.3, z_g is $\mathcal{N}_{\mathrm{SR}}$-null-additive over $\mathbf{E}$. Then, by Lemma 4.4, $\langle z_g \upharpoonright [b(n), b(n+1))\rangle_{n\in\omega}$ is $\mathcal{T}_{\mathrm{tt}}$-traceable over $\mathbf{E}$. Consequently, g is traceable over $\mathbf{E}$.

Conversely, suppose that every $g \leq_{\mathcal{F}} A$ is $\mathcal{T}_{\mathrm{tt}}$-traceable over $\mathbf{E}$. Let $N(A) \in \mathcal{N}$ be an $\mathcal{N}_{\mathrm{SR}}$-null set $\mathcal{F}$-uniformly in A over $\mathbf{E}$, that is, there is $p \leq_{\mathcal{F}} A$ such that $\rho_{\mathrm{SR}}(p) = N(A)$. By Lemma 5.2, we can continuously find q such that $\rho'_{\mathrm{SR}}(q) = \rho_{\mathrm{SR}}(p) = N(A)$. Then, we have $q \leq_{\mathcal{F}} A$, since the inclusion map $\mathcal{C} \hookrightarrow \mathcal{F}$ is $\mathbf{E}$-coded. Since $q \leq_{\mathcal{F}} A$ is traceable over $\mathbf{E}$, it is easy to see that q is bounded by a function in $\mathbf{E}$. Hence, q can be identified with a real $z_q \in 2^\omega$ as before. Then, obviously, q is $\star$-traceable over $\mathbf{E}$, and therefore, q is uniform low for $\mathcal{N}_{\mathrm{SR}}$-randomness over $\mathbf{E}$ by Lemma 4.3. Clearly, the map $\rho'_{\mathrm{SR}} : \omega^\omega \to \mathcal{N}$ is contained in $\mathcal{C}^{\mathbf{E}}_{\mathrm{tt}}$. Hence, $N(A) = \rho_{\mathrm{SR}}(q)$ is covered by an $\mathcal{N}_{\mathrm{SR}}$-null set over $\mathbf{E}$ by uniform lowness of q. ☐

This for instance implies that a real is lowness for Δ^1_2-quasirandomness (where a real is Δ^1_2-quasirandom iff it avoids all Lebesue null Δ^1_2-coded Borel set) is equivalent to Δ^1_2-traceability, a real A is lowness for Δ^1_1-Schnorr randomness w.r.t. continuous relativization iff every $g \leq_{\mathrm{hT}} A$ is Δ^1_1-traceable, where $\leq_{\mathrm{hT}}$ is higher-Turing reducibility (see [5]), etc.

It is known that Π^1_1-traceability is equivalent to Δ^1_1-traceability (see [57]). Generally, $\mathbf{E}$ is generated by Spector pointclasses, then one can see that $\mathcal{T}_{\mathrm{semi}}$-traceability is equivalent to $\mathcal{T}_{\mathrm{tt}}$-traceability over $\mathbf{E}$. We say that $\mathbf{E}$ is *locally Spector* if it is KS-based by a collection $(\Gamma_u)_{u\in\Lambda}$ of Spector pointclasses (see Definition 2.17).

Lemma 5.4. *Suppose that $\mathbf{E}$ is a locally Spector set, and the composition operation*

$$\circ : \mathcal{F}_{\mathbf{\Pi}^1_1}(\subseteq\omega^\omega, \omega^\omega) \times \mathcal{F}(\subseteq\omega^\omega, \omega^\omega) \to \mathcal{F}(\subseteq\omega^\omega, \omega^\omega)$$

is partial computable. Then, $(\mathcal{T}_{\mathrm{semi}}, \tilde{\mathsf{Q}}, \mathcal{F}, \mathbf{E})$-traceability is equivalent to $(\mathcal{T}_{\mathrm{tt}}, \tilde{\mathsf{Q}}, \mathcal{F}, \mathbf{E})$-traceability.

Proof. Suppose that x is $(\mathcal{T}_{\rm semi}, \tilde{\rm Q}, \mathcal{F}, \mathbf{E})$-traceable. Then for any $\Phi \in \mathcal{F}^{\mathbf{E}_a}$, there is $(T_n) \in \mathcal{T}_{\rm semi}^{\mathbf{E}_b}$ such that $\Phi(x)(n) \in T_n$ for all $n \in \omega$. Let $\mathbf{E}_c$ be a Spector set which is an amalgamation of $\mathbf{E}_a$ and $\mathbf{E}_b$, and Γ_c be a corresponding Spector pointclass. Now consider a Γ_c-norm φ on the Γ_c-semicoded set $Z = \{(k,n) \in \omega \times \omega : k \in T_n\}$. Given $g \in \omega^\omega$, if $(g(n), n) \in Z$ for all n, define $\Psi(g) \in {\rm WO}^\omega$ representing $(\varphi(g(n), n))_{n\in\omega}$. Then Ψ is partial continuous and $\mathbf{E}_c$-semicoded. One can define the "halting probability" $\Omega_{\mathbf{E}_c}$ as an analogy of Chaitin's Ω via the space $\mathcal{C}_{\rm semi}^{\mathbf{E}_c}(2^{<\omega})$, and show that $\Omega_{\mathbf{E}_c}$ is Martin-Löf random over $\mathbf{E}_c$ and not i.o. $\mathcal{T}_{\rm semi}$-traceable over $\mathbf{E}_c$. Suppose that $\sup \Psi(g)(n) \geq \gamma$, where γ is the supremum of all Γ ordinals. Let L_e be the e-th Γ_c linear order on ω. Then $n \notin \Omega_{\mathbf{E}_c}$ iff for every e and n, either L_e is not embedded into $\Psi(g,n)$ or $\theta(e)$ is not embedded into L_e, where θ is a Γ_c-norm on $\Omega_{\mathbf{E}_c}$. Thus, there is a $\Theta \in \mathcal{F}_{\mathbf{\Pi}^1_1}$ such that $\Theta(g) = \Omega_{\mathbf{E}_c}$.

Therefore, $\sup \Psi(\Phi(x))(n) < \gamma$, otherwise $\Omega_{\mathbf{E}_c} = \Theta(\Phi(x)) \leq_{\mathcal{F}}^{\mathbf{E}_c} \Phi(x) \leq_{\mathcal{F}}^{\mathbf{E}_c} x$ is $\tilde{\rm Q}$-often $\mathcal{T}_{\rm semi}$-traceable over $\mathbf{E}_c \supseteq \mathbf{E}_a$. Let δ be a Δ-ordinal such that $\sup \Psi(\Phi(x))(n) < \delta$, and define $T_n[\delta] = \{k \in T_n : \varphi(k,n) < \delta\}$. Then $(T_n[\delta])_{n\in\omega}$ is Δ-coded (i.e., Γ-coded in $\mathcal{T}_{\rm tt}$). □

This for instance implies that lowness for ITTM-Schnorr randomness, lowness for (ITTM-Martin-Löf, ITTM-Schnorr)-randomness, ITTM-computable traceability, and ITTM-semicomputable traceability are all equivalent.

5.4. ${\rm SR}^{\mathcal{M}}$ **Sets.** We next effectivize Bartoszyński-Judah's characterization [2] of ${\rm SR}^{\mathcal{M}}$ sets.

Theorem 5.5. *A set $A \subseteq \mathbb{N}$ is $\mathcal{F}$-uniformly low for Cohen quasigenericity over $\mathbf{E}$ if and only if every $g \leq_{\mathcal{F}} A$ is $\mathbf{E}$-often $\mathbf{E}$-traceable.*

Proof. Suppose that $A \subseteq \mathbb{N}$ is $\mathcal{F}$-uniformly low for Cohen quasigenericity over $\mathbf{E}$. Let $g \leq_{\mathcal{F}}^{\mathbf{E}} A$. Suppose for the sake of the contradiction that g is not $\mathbf{E}$-bounded. However, the unboundedness of g implies that $M_g^{\vec{0}}$ is not meager over $\mathbf{E}$ (hence, not contained in $\mathcal{E}$ over $\mathbf{E}$), where $A \mapsto M_g^{\vec{0}}$ is $\mathcal{F}$-uniform. Therefore, g is bounded by an $\mathbf{E}$-function h. Then, as in Theorem 5.3, g is coded as a real $z_g \in 2^\omega$. Since $z_g \leq_{\mathcal{F}} g \leq_{\mathcal{F}} A$, z_g is meager-additive over $\mathbf{E}$ by Lemma 4.13. Therefore, by Lemma 4.12, z_g is $\mathbf{E}$-often $\mathbf{E}$-traceable.

For the converse direction, we can use the similar argument as in Theorem 5.3 since we have the total representation that is equivalent to $\rho_{\rm W1G}$ by Lemma 5.2. □

This for instance implies that lowness for ITRM-Cohen-quasigenericity is equivalent to ITRM-computably-often ITRM-computable traceability.

We may define the notion of a ${\rm SR}^{\mathcal{E}}$ set by the straightforward way. Then, we can give a new result concerning ${\rm SR}^{\mathcal{E}}$ sets and the Borel image of $\mathcal{E}$-additive sets.

Theorem 5.6. *A set $A \subseteq \mathbb{N}$ is $\mathcal{F}$-uniformly low for Kurtz randomness over $\mathbf{E}$ if and only if every $g \leq_{\mathcal{F}} A$ is $\mathbf{E}$-often traceable over $\mathbf{E}$.*

Proof. For the "only if" direction, similar as the proof of Theorem 5.5. For the converse direction, we can also use the similar argument as in Theorem 5.3 since we have the total representation that is equivalent to $\rho_{\rm WR}$ by Lemma 5.2. □

This for instance implies that a real A is lowness for Δ^1_1-Kurtz randomness w.r.t. continuous relativization iff every $g \leq_{\rm hT} A$ is Δ^1_1-often Δ^1_1-traceable.

6. Other Results

6.1. **The Kučera-Gács Theorem and Strong Measure Zero.** The Kučera-Gács theorem says that every real is computable in a Martin-Löf random real (see [24, Theorem 8.3.2]). Hjorth-Nies [36] showed a higher analog of the Kučera-Gács theorem: Every real is ${\rm wtt}(\Pi^1_1)$-reducible to a Π^1_1-Martin-Löf random real. In this section, we show the following abstract version of the Kučera-Gács Theorem.

Theorem 6.1 (Generalized Kučera-Gács Theorem)**.** *Suppose that Γ is a Kleene-or-Spector pointclass. Then, every real is* ${\rm wtt}(\Gamma)$*-reducible to a Γ-Martin-Löf random real.*

Binns [8] studied the effective perfect set property, and clarified a mechanism behind the Kučera-Gács theorem. The first author (see [32]) introduced the notion of *effectively strongly measure zero* to obscure the point of Binns' idea.

Definition 6.2. Let $\mathbf{E}$ be a locally KS set. A set $P \subseteq 2^\omega$ is $\mathbf{E}$-*rapidly perfect* if there exists a function $f \in \omega^\omega \cap \mathbf{E}$, for every $\sigma \in 2^n$, if $P \cap [\sigma] \neq \emptyset$, there are at least two incomparable strings $\tau_0, \tau_1 \in 2^{f(n)}$ extending σ such that $P \cap [\tau_0] \neq \emptyset$ and $P \cap [\tau_1] \neq \emptyset$.

The following is a generalization of the theorem of Binns [8] and Higuchi-Kihara [32]. Recall that a closed set $P \subseteq 2^\omega$ is $\mathbf{E}$-*semicoded* if there is $p \in \mathbf{E} \cap \mathcal{O}(2^{<\omega})$ (via an effective bijection between ω and $2^{<\omega}$) such that $P = 2^\omega \setminus \bigcup_{\sigma \in p}[\sigma]$. By effective compactness of P, it is equivalent to that the complement of $\mathrm{Ext}(P) := \{\sigma \in 2^{<\omega} : [\sigma] \cap P \neq \emptyset\}$ is in $\mathbf{E}$.

Theorem 6.3. *Let $\mathbf{E}$ be a countable locally KS set. Then, the following are equivalent for an $\mathbf{E}$-semicoded closed set $P \subseteq 2^\omega$.*

(i) *P is not strong measure zero over $\mathbf{E}$.*
(ii) *P has an $\mathbf{E}$-rapidly perfect subset.*
(iii) *There is a partial uniformly continuous function $\Gamma \in \mathcal{C}^{\mathbf{E}}_{\mathrm{uf}}$ such that $\Gamma[P \cap \mathrm{dom}(\Gamma)] = 2^\omega$.*
(iv) *Every real is wtt($\mathbf{E}$)-reducible to an element of P.*

Proof. (1)⇒(2): Higuchi-Kihara [32, Theorem 43] shows that if $\{k_i\}_{i\in\mathbb{N}}$ witnesses that P is not of strong measure zero, then $n \mapsto k_{F(n+1)}$ witnesses that a subset of P is rapidly perfect, where $F : \omega \to \omega$ is defined recursively by $F(0) = 0$ and $F(n+1) = F(n) + 2^{k_{F(n)}}$. Clearly, if such a witness $\{k_i\}_{i\in\mathbb{N}}$ is $\mathbf{E}$-coded, so is $n \mapsto k_{F(n+1)}$.

(2)⇒(3): Let $Q \subseteq P$ be an $\mathbf{E}$-rapidly perfect $\mathbf{E}$-semicoded closed set. Then, there is an order $h \in (\omega^\omega)^{\mathbf{E}}$ such that every extendible $\sigma \in \mathrm{Ext}(Q)$ of length $h(n)$ has two extensions of length $h(n+1)$ in $\mathrm{Ext}(Q)$. We define a wtt($\mathbf{E}$)-reduction Γ as follows. For strings $\tau \in 2^{h(n)}$ and $\sigma \in 2^n$, put $(\tau, \sigma) \in \Gamma$ if and only if for all $i < n$, either

$$\sigma(i) = 0 \;\&\; \forall \rho \in 2^{h(i+1)}\ [(\rho \prec \tau \upharpoonright h(i) \;\&\; \rho <_{\mathrm{left}} \tau \upharpoonright h(i+1)) \;\to\; \rho \notin \mathrm{Ext}(Q)],$$
$$\text{or } \sigma(i) = 1 \;\&\; \forall \rho \in 2^{h(i+1)}\ [(\rho \prec \tau \upharpoonright h(i) \;\&\; \rho >_{\mathrm{left}} \tau \upharpoonright h(i+1)) \;\to\; \rho \notin \mathrm{Ext}(Q)],$$

where $\rho <_{\mathrm{left}} \tau$ denotes that ρ is left to τ. Clearly, $\Gamma \in \mathcal{P}(2^{<\omega} \times 2^{<\omega})$ is in $\mathbf{E}$. Since h witnesses rapid perfectness of Q, this set Γ generates an $\mathbf{E}$-semicoded partial function on 2^ω whose modulus h of uniform continuity is $\mathbf{E}$-coded. It is not hard to see that for every $y \in 2^\omega$, there exists $x \in Q$ such that $\Gamma(x) = y$.

(3)⇒(4): Obvious.

(4)⇒(1): Since $\mathbf{E}$ is countable, there is a real x which is not i.o. $\mathcal{T}_{\mathrm{semi}}$-$\star$-traceable over $\mathbf{E}$. By our assumption, there is $y \in P$ such that $x \leq_{\mathrm{wtt}(\mathbf{E})} y$. Then, there is a partial $\mathbf{E}$-semicoded uniformly continuous function Ψ with an $\mathbf{E}$-coded modulus $g \in (\omega^\omega)^{\mathbf{E}}$, i.e., $\Psi^{y\upharpoonright g(n)} \upharpoonright n = x \upharpoonright n$. Suppose for the sake of contradiction that P is strongly measure zero over $\mathbf{E}$. Then, given $u \in (\omega^\omega)^{\mathbf{E}}$, there exists a partial $\mathbf{E}$-semicoded map $p :\subseteq \omega \to 2^{<\omega}$ such that $y \upharpoonright g(u(n)) = p(n)$ for infinitely many $n \in \omega$. Thus, the partial $\mathbf{E}$-semicoded sequence $\langle \Psi^{p(n)} \upharpoonright u(n) \rangle$ hits $x \upharpoonright u(n)$ infinitely often. Then, x is i.o. $\mathcal{T}_{\mathrm{semi}}$-$\star$-traceable over $\mathbf{E}$. □

Proof of Theorem 6.1. Let Γ be a Kleene-or-Spector pointclass. Since Γ is ω-parametrized, there is an optimal Γ-machine, and then one can define the Γ-Kolmogorov complexity K_Γ. By the standard argument, it is not hard to see that x is Γ-Martin-Löf random if and only if x is incompressible w.r.t. K_Γ, that is, $K_\Gamma(x \upharpoonright n) \geq n - O(1)$. Clearly, the set of 1-incompressible strings, $T = \{\sigma : (\forall \tau \prec \sigma)\ K_\Gamma(\tau) \geq |\tau| - 1\}$, forms an co-$\mathbf{E}_\Gamma$ tree. Moreover, the measure of the $\mathbf{E}_\Gamma$-closed set $[T]$ is at least $1/2$ by the prefix-free condition of our machine. Therefore, by Theorem 6.3, every real is wtt(Δ)-reducible to an element $x \in [T]$. Note that such an $x \in [T]$ is Γ-Martin-Löf random since K_Γ is defined via an optimal machine. □

Consequently, the Kučera-Gács Theorem holds for Π^1_1-Martin-Löf randomness, Σ^1_2-Martin-Löf randomness, ITTM-Martin-Löf randomness, etc. However, the Kučera-Gács Theorem fails for non-principal KS sets.

Proposition 6.4. *The Kučera-Gács Theorem fails for arithmetical randomness.*

Proof. If x is arithmetically random, x should be n-random for every $n \in \omega$. However, if x is $(n+1)$-random, then we have $\emptyset^{(n)} \not\leq_T x^{(n-1)}$ (see [24, Theorem 8.14.1]). Therefore, $\emptyset^{(\omega)}$ is not arithmetically reducible to an arithmetically random real x. □

Proposition 6.5. *The Kučera-Gács Theorem fails for ITRM-Martin-Löf randomness.*

Proof. It suffices to show that if x is ITRM-Martin-Löf random, then $\omega_i^{\mathrm{CK},x} = \omega_i^{\mathrm{CK}}$ for all $i \in \omega$. We first note that if x is ITRM-Martin-Löf random, then x avoids all null $\Delta^1_1(z)$ sets for $n \in \omega$ and $z \in L_{\omega_\omega^{\mathrm{CK}}} \cap 2^\omega$, where $\omega_\alpha^{\mathrm{CK}}$ is the α-th ordinal which is admissible or limit of admissibles. Carl-Schlicht [17] showed that $X_i = \{x \in 2^\omega : \omega_1^{\mathrm{CK},x\oplus c(i)} = \omega_1^{\mathrm{CK},c(i)}\}$ is co-null $\Sigma^1_1(c(i))$ set, where $c(i)$ is the $<_L$-least code of the i-th admissible ordinal ω_i^{CK}. By relativizing Chong-Nies-Yu's theorem [19], every $\Delta^1_1(\mathcal{O}^{c(i)})$-random real is also $\Pi^1_1(c(i))$-random, that is, it avoids all null $\Pi^1_1(c(i))$ sets, where $\mathcal{O}^{c(i)}$ denotes Kleene's $\mathcal{O}$ relative to $c(i)$. Since $\mathcal{O}^{c(i)} \in L_{\omega_{i+2}^{\mathrm{CK}}}$, any ITRM-Martin-Löf random real should be $\Delta^1_1(\mathcal{O}^{c(i)})$-random; hence, $\Pi^1_1(c(i))$-random. Therefore, any ITRM-Martin-Löf random is contained in X_i for all $i \in \omega$. Hence, we have $\omega_i^{\mathrm{CK},x} = \omega_i^{\mathrm{CK}}$ for all $i \in \omega$ (see Carl-Schlicht [17] for the detail). □

6.2. **Distribution of Trivial Reals.** In this section, we will discuss about triviality (i.e., null-additivity) in generalized computability theory. Throughout this section, we assume that Γ is a Kleene-or-Spector pointclass. We say that $A \subseteq \omega$ is Γ-*trivial* if it is K_{semi}-trivial over $\mathbf{E}_\Gamma$, and Δ-*trivial* if it is $K_{\mathrm{tt}\lambda}$-trivial over $\mathbf{E}_\Gamma$. For instance, Δ^0_1-triviality is equivalent to Σ^0_1-triviality is equivalent to K-triviality, and Π^1_1-triviality is known as $\underline{K}$-triviality in higher randomness theory. Recall that Δ-triviality is equivalent to Δ-coded-null additivity (Theorems 3.15 and 4.2).

Proposition 6.6. *There are continuum many Δ-trivial reals, whereas there are only countably many Γ-trivial reals.*

Proof. Franklin-Stephan [28] observed that any subset of a dense immune Σ^0_1 set is a universally indifferent for Schnorr randomness. We say that A is Δ-*dense-immune* if it is infinite, and its principle function p_A is Δ-dominant, that is, for every $g \in \omega^\omega \cap \Delta$, $p_A(n) \geq g(n)$ for almost all n. It is easy to see that there is a Δ-dense immune co-Γ subset of ω since Γ is ω-parametrized. Now, consider the set of all subsets of such a Δ-dense-simple co-Γ subset of ω (see [28]).

It is easily to verify the latter assertion by a straightforward modification of Chaitin's counting argument (see [57, Theorem 5.2.4]). Indeed, every Γ-trivial set is reducible to a universal Γ set. □

Solovay [71] showed that a noncomputable K-trivial Σ^0_1 set exists, and Hjorth-Nies [36] showed the analogous results at the hyperarithmetical level. The straightforward modification of the cost function method (see [57, Section 5.3]) easily implies the existence of a Γ-trivial Γ set $A \notin \Delta$.

We will see that Δ-triviality and Γ-triviality are incomparable.

Theorem 6.7. *Suppose that Γ is a Kleene-or-Spector pointclass.*

(i) *There exists a Γ set $A \subseteq \omega$ that is Γ-trivial, but not Δ-trivial.*
(ii) *There exists a Γ set $A \subseteq \omega$ that is Δ-trivial, but not Γ-trivial.*

Lemma 6.8. *Γ-triviality is downward closed under* wtt(Γ)*-reducibility, and Δ-triviality is downward closed under* tt(Γ)*-reducibility.*

Proof. Suppose that A is Γ-trivial. Let Ψ be a wtt(Γ)-reduction witnessing $\Psi(A) = B$ with modulus $u \in \Delta$. Then, $K^{\mathbf{E}_\Gamma}_{\mathrm{semi}}(B \upharpoonright n) \leq^+ K^{\mathbf{E}_\Gamma}_{\mathrm{semi}}(A \upharpoonright u(n)) \leq^+ K^{\mathbf{E}_\Gamma}_{\mathrm{semi}}(u(n)) \leq^+ K^{\mathbf{E}_\Gamma}_{\mathrm{semi}}(n)$. Thus, B is Γ-trivial.

Suppose that A is Δ-trivial. Let Ψ be a tt(Γ)-reduction witnessing $\Psi(A) = B$ with modulus $u \in \Delta$. Given $\mathbf{E}_\Gamma$-semicoded machine $\varphi \in \mathcal{C}_{\mathrm{tt}\lambda}(2^{<\omega})$, clearly $\psi = \{\langle \sigma, \Psi(\tau) \upharpoonright n\rangle : (\exists n \in \omega)\ \tau \in 2^{u(n)}\}$ is $\mathbf{E}_\Gamma$-semicoded w.r.t. the representation $\rho_{\mathrm{tt}\lambda}$. Hence, ψ is $\mathbf{E}_\Gamma$-semicoded, and $K_\psi(B \upharpoonright n) \leq K_\varphi(A \upharpoonright u(n))$. Therefore, B is Δ-trivial. □

Franklin-Stephan [28, Theorem 5.2] showed that the class of Schnorr trivial sets is not closed under *wtt*-reducibility. Recall that Kleene's recursion theorem holds for any Kleene-or-Spector class (see [25, 55]). Formally, let $\{\varphi_e\}_{e\in\omega}$ be a Γ enumeration of all partial functions with Γ graph. Then, for every $h \in \omega^\omega \cap \Delta$, there is a fixed point r such that $\varphi_{h(r)} = \varphi_r$. Thus, it is straightforward to show the following analog of Franklin-Stephan's theorem.

Lemma 6.9. *For every Γ set $A \subseteq \omega$, if $A \notin \Delta$, there exists a Γ set $B \subseteq \omega$ such that $B \equiv_{\mathrm{wtt}(\Gamma)} A$ and B is not Δ-trivial.* □

Lemma 6.10. *There exists a co-Γ set $A \subseteq \omega$ such that A is Δ-dense-immune and A is not Γ-trivial.*

Proof. $K^{\mathbf{E}_\Gamma}_{\rm semi}(n) \leq n + \log n + c \leq 2n$. Approximate the principal function p_A. If $K^{\mathbf{E}_\Gamma}_{\rm semi}(p_{A,\beta}\upharpoonright n)[\beta] \leq 2n$ for some stage $\beta < \alpha$, make $p_A(n-1)$ bigger. Moreover, make $p_A(n)$ bigger than $\varphi^\Gamma_e(n)$ for every $e \leq n$, where φ^Γ_e is a partial function constructed from the e-th Γ set. Note that if A is Γ-trivial, then by analyzing the usual proof of the assertion that K-triviality implies lowness for K, we have $K^{\mathbf{E}_\Gamma}_{\rm semi}(p_A(n)) \leq K^{\mathbf{E}_\Gamma}_{\rm semi}(n)+O(1)$. However, this implies $K^{\mathbf{E}_\Gamma}_{\rm semi}(n) > 2n - O(1)$. Hence, A cannot be Γ-trivial. □

Lemmata 6.8, 6.9 and 6.10 clearly imply Theorem 6.7. This for instance implies that Π^1_1-triviality and Δ^1_1-triviality are incomparable even for Π^1_1 subsets of ω.

7. Question

By Lemma 3.4, A is Martin-Löf null-additive if and only if $A \triangle Z$ is Martin-Löf random whenever Z is Martin-Löf random. It is natural to ask whether the Schnorr version of this characterization holds.

Problem 7.1. *Let $A \subseteq \mathbb{N}$. Suppose that $A \triangle Z$ is Schnorr random whenever Z is Schnorr random. Then, is A Schnorr null-additive?*

If A is low for Martin-Löf randomness, then clearly, A is Martin-Löf null-additive. Hence, K-triviality implies Martin-Löf null-additivity, since K-triviality is equivalent to lowness for Martin-Löf randomness. Thus, it is natural to ask if the converse holds.

Problem 7.2. *Let A be a subset of $\mathbb{N}$. Suppose that $A \triangle Z$ is Martin-Löf random for every Martin-Löf random set $Z \subseteq \mathbb{N}$. Then, is A necessary to be K-trivial?*[1]

Recall that K-triviality implies Δ^0_2. Hence, only countably many K-trivial sets exist. However, we do not know even whether the class of the Martin-Löf null-additive sets is countable.

Problem 7.3. *Does there exist a Martin-Löf null-additive set which is not computable in $\emptyset'$? Or even, do there exist uncountably many Martin-Löf null-additive sets?*[2]

As mentioned in Section 4.5, by straightforward effectivization of the equivalence $\mathrm{add}^\star(\mathcal{E},\mathcal{N}) = \mathrm{cov}^\star(\mathcal{M})$, we have a covering characterization of $\mathrm{Low}^\star(\mathrm{SR},\mathrm{WR})$, that is, $A \in \mathrm{Low}^\star(\mathrm{SR},\mathrm{WR})$ if and only if $2^\omega \subseteq \{A\} + M$ for every computable meager set $M \subseteq 2^\omega$. Now, one can ask whether a covering characterization of $\mathrm{Low}^\star(\mathrm{MLR},\mathrm{WR})$ exists.

Problem 7.4. *Do there exist a class of meager sets $\mathcal{M}^* \subseteq \mathcal{M}$ and a class of reals $\mathcal{C} \subseteq 2^\omega$ such that for every $A \in 2^\omega$, $A \in \mathrm{Low}^\star(\mathrm{MLR},\mathrm{WR})$ if and only if $\mathcal{C} \subseteq \{A\} + M$ for every $M \in \mathcal{M}^*$?*

Recall from Theorem 2.49 that Π^1_1-$\star$-traceability is equal to higher-anticomplexity (i.e., $\{\mathsf{Q}_{\rm cof}\}$-often $K_{\rm semi}$-compressibility over $\mathbf{E}_{\Pi^1_1}$). One can also show that $(\mathcal{T}_{\rm semi}, \tilde{Q}; \mathcal{C}, \mathbf{E})$-traceability is equal to $\tilde{Q}$-often $K_{\rm semi}$-autocompressibility over $\mathbf{E}$, e.g., the higher-Turing degree of a real x is Π^1_1-traceable iff x is higher-auto-anticomplex. It is shown in [44, Theorem 5.3] that a real is computably traceable iff it is totally auto-anticomplex. However, the standard proof requires a *time-trick* (in the sense of [5]); therefore, the proof is applicative only for ω-normed pointclasses.

Problem 7.5. *If a real x is higher-totally-auto-anticomplex, then is it true that the higher-Turing degree of x is Δ^1_1-traceable? Formally speaking, does $\{\mathsf{Q}_{\rm cof}\}$-often $K_{{\rm tt}\lambda}$-autocompressibility over $\mathbf{E}_{\Pi^1_1}$ imply $(\mathcal{T}_{\rm tt}, \{\mathsf{Q}_{\rm cof}\}; \mathcal{C}, \mathbf{E}_{\Pi^1_1})$-traceability?*

In algorithmic randomness theory, there are many results stating that lowness for randomness is equal to lowness for tests. Currently we do not have a complete result on the equivalence of lowness for randomness and tests at non-algorithmic levels.

Problem 7.6. *Can Theorem 3.10 be extended to good$_2$-pairs? Do we really need countability and principality of $\mathbf{E}$ to show Theorem 3.10?*

[1] Kuyper and Miller recently announced that this problem has an affirmative answer.

[2] An affirmative answer to Problem 7.2 implies negative answers to these problems.

It is also important to develop separation techniques for uniform lowness properties. For instance, we can ask the following.

Problem 7.7. *Does there exist a real which is low for Δ^1_1-randomness w.r.t. continuous relativization, but not low for Δ^1_1-randomness? Formally speaking, does $(\mathcal{N}_{\rm SR},\mathcal{N}_{\rm SR};\mathcal{C},\mathbf{E}_{\Pi^1_1})$-lowness imply $(\mathcal{N}_{\rm SR},\mathcal{N}_{\rm SR};\mathcal{F}_{\mathbf{\Pi}^1_1},\mathbf{E}_{\Pi^1_1})$-lowness?*

For instance, Sacks forcing generates a Δ^1_1-traceable real. Such a well known technique has the *continuous reading of names* [78], that is, if a real is h-reducible to a Sacks real γ over $L_{\omega_1^{\rm CK}}$, then it is reducible to γ via a Π^1_1-semi-coded continuous function. To solve Problem 7.7, we need a new proof technique which has no continuous reading of names. Hence, Problem 7.7 and its variations are quite important to develop proof techniques in higher recursion theory, and even in set theory.

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