MUCHNIK DEGREES AND MEDVEDEV DEGREES OF THE RANDOMNESS NOTIONS

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Abstract. The main theme of this paper is computational power when a machine is allowed to access random sets. The computability depends on the randomness notions and we compare them by Muchnik and Medvedev degrees. The central question is whether, given an random oracle, one can compute a more random set. The main result is that, for each Turing functional, there exists a Schnorr random set whose output is not computably random.

1. Introduction

In mathematical logic or theoretical computer science, a function $f :\subseteq \omega \to \omega$ is *computable* if the function f is computable by a Turing machine. Computability on natural numbers is formalized by a few ways, that have been proved to be equivalent. By the Church-Turing thesis, we believe that this is the correct definition of computability. The definition naturally extends to a function $f :\subseteq 2^\omega \to 2^\omega$.

When one is writing a code in a programming language, one may use a random generator and may say that a function is computable even if the function uses a random generator. We formalize computational power of functions f on Cantor space when the functions are allowed to access random sets. In computational complexity theory, there are some such classes such as BPP and RP. In this paper, we focus on computability theory.

One old answer is the following. Suppose a set X is Turing computable with an oracle in a class of random sets. The set of random sets (for any reasonable randomness notion) has measure 1. Thus, $\{Y \in 2^\omega : Y \geq_T X\}$ has measure 1. A classical theorem says that, if a set $X \in 2^\omega$ is not computable, then $\{Y \in 2^\omega : Y \geq_T X\}$ has Lebesgue measure 0 [6]. Hence, X should be an computable, which means that it is already computable without a random oracle. This is the case with only one solution.

When multiple solutions are allowed, the results are very different. The computability with random access can be formalized as mass problems. Each element $X \in 2^\omega$ can be seen as a candidate of solutions and we identify the set of solutions and a problem. Consider two problems $P, Q \subseteq 2^\omega$. If each solution $f \in Q$ can compute a solution $g \in P$, then we can say that constructing a solution of P is not more difficult than constructing a solution of Q. The induced degree of P will represent computational power when allowed to access an element of P. When the computation is uniform, we say that P is *Medvedev (or strongly) reducible to* Q and it is denoted by $P \leq_s Q$. When the computation can be nonuniform, P is *Muchnik (or weakly) reducible to* Q and denoted by $P \leq_w Q$. For a formal definition, see Definition 2.1.

Many randomness notions have been studied in the theory of algorithmic randomness. Each randomness notion is a subset of Cantor space, so we can directly study its degree. If a randomness notion P is weaker than Q ($P \supseteq Q$), then it straightforwardly implies $P \leq_s Q$ and $P \leq_w Q$. In other words, for any $P, Q \subseteq 2^\omega$, we have

$$P \supseteq Q \Rightarrow P \leq_s Q \Rightarrow P \leq_w Q.$$

So, the real problem is whether each reduction is strict. In other words, we ask whether one can compute a more random set from a given random set.

The following is (a part of) well-known hierarchy of randomness notions:

$$\mathrm{WR} \supsetneq \mathrm{SR} \supsetneq \mathrm{CR} \supsetneq \mathrm{MLR} \supsetneq \mathrm{DiffR} \begin{array}{c} \supsetneq \mathrm{W2R} \supsetneq \\ \supsetneq \mathrm{DemR} \supsetneq \end{array} \mathrm{2R}.$$

The abbriviations of randomness notions above will be explained in the next section. The following is the structure of Muchnik and Medvedev degrees, and the summary of this paper:

$$\mathrm{WR} <_w \mathrm{SR} \equiv_w \mathrm{CR} <_w \mathrm{MLR} \equiv_w \mathrm{DiffR} \begin{array}{c} <_w \mathrm{W2R} <_w \\ <_w \mathrm{DemR} <_w \end{array} \mathrm{2R}, \tag{1}$$

$$\mathrm{SR} <_s \mathrm{CR}, \quad \mathrm{MLR} <_s \mathrm{DiffR}. \tag{2}$$

The main result is $\mathrm{SR} <_s \mathrm{CR}$ and will be proved in Theorem 3.10.

A related work can be found in Simpson [22], which mainly studied the Muchnik degrees of Π^0_1-class and the relation with the Muchnik degree of the class of all ML-random sets. A similar question can be asked in the context of Weihrauch degrees [3].

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The rest of the paper is devoted to the proofs of these results.

2. Preliminaries

2.1. **Computability theory.** We fix some notations. We denote the set of finite binary strings by $2^{<\omega}$, and the set of infinite binary sequences by 2^ω. We often identify an infinite sequence $X \in 2^\omega$ with a set of natural number $Y \subseteq \omega$ by $X(n) = 1 \iff n \in Y$. The *jump* of a sequence A is denoted by A' and is defined by $A' = \{n \in \omega : \Phi_n^A(n)\downarrow\}$, there Φ_n is the n-th Turing machine.

For computability theory, see some textbooks such as [5, 23].

We define Muchnik and Medvedev reducibility on 2^ω because we are interested in only randomness notions.

Definition 2.1 (Muchnik [16], Medvedev [12])**.** Let $P, Q \subseteq 2^\omega$.

(i) We say that P is *Muchnik reducible (or weakly reducible)* to Q, denoted by $P \leq_w Q$, if, for every $f \in Q$, there is an element $g \leq_T f$ in P.
(ii) We say that P is *Medvedev reducible (or strongly reducible)* to Q, denoted by $P \leq_s Q$, if there is a Turing functional Φ such that $\Phi^f \in P$ for every $f \in Q$.

A rough idea is that $P \leq_w Q$ if each solution in Q can Turing compute a solution in P, but the reduction need not be uniform. In contrast $P \leq_s Q$ if one Turing reduction Φ can compute a solution in P from a solution in Q uniformly. The reductions $\leq_w$ and $\leq_s$ are pre-orders. We write $P \equiv_w Q$ to mean $P \leq_w Q$ and $Q \leq_w P$, and $P \equiv_s Q$ similarly. Muchnik degrees and Medvedev degrees are equivalent classes derived from the equivalent relations $\equiv_w$ and $\equiv_s$, respectively.

2.2. **The theory of algorithmic randomness.** We review some definitions or characterizations from the theory of algorithmic randomness. For more details, see monographs [18, 7].

Cantor space is the space 2^ω equipped with the topology generated by the cylinder sets $[\sigma] = \{X \in 2^\omega : \sigma \prec X\}$ for $\sigma \in 2^{<\omega}$, where $\prec$ is the prefix relation.

We define $X \oplus Y = \{2n : n \in X\} \cup \{2n+1 : n \in Y\}$. We usually consider the fair-coin measure λ on 2^ω.

An open set V is *c.e.* if $V = \bigcup_{\sigma \in S}[\sigma]$ for a computable set S. A Martin-Löf test (ML-test) is a sequence $\{U_n\}_{n\in\omega}$ of uniformly c.e. open sets such that $\lambda(U_n) \leq 2^{-n}$ for all n. A sequence $X \in 2^\omega$ is *ML-random* if $X \notin \bigcap_n U_n$ for each ML-test. The class of ML-random sequences is denoted by MLR. Note that MLR is a subset of 2^ω. We compare this class with other randomness notions in Muchnik degrees and Medvedev degrees.

We denote the classes of Kurtz random sets, Schnorr random sets, computably random sets, difference random sets, weakly 2-random sets, Demuth random sets, and 2-random sets by WR, SR, CR, DiffR, W2R, DemR, and 2R, respectively. In the proof we need only their characterizations or some properties.

The randomness notion depends on the measure λ. If one replaces λ with a computable measure ν in the definition of ML-randomness, then we can define ML-randomness with respect to ν, or ν-ML-randomness. Other randomness notions with respect to a computable measure can be defined similarly.

3. Proofs

3.1. **Separations in Muchnik degrees.** Here, we show equivalences and separations of randomness notions in Muchnik degrees. In the proof we make use of some notions in Turing degrees.

We start from the proof of CR $<_w$ MLR. The proof is probably simplest and it is appropriate to explain the intuition from this. We look for a Turing degree $\mathbf{a}$ that contains a computably random set but no Turing degree below $\mathbf{a}$ contains a ML-random set. We use the notion of highness to prove that the degree contains a computably random set, and the one of minimality to prove that the degrees do not contain any ML-random set.

A Turing degree $\mathbf{a} > \mathbf{0}$ is *minimal* if no degree $\mathbf{b}$ satisfies $\mathbf{0} < \mathbf{b} < \mathbf{a}$. We say that a Turing degree $\mathbf{a}$ is ML-random if it contains a ML-random set.

Lemma 3.1. *No ML-random degree is minimal.*

Proof. Assume, for a contradiction, that $Z = X \oplus Y$ is a ML-random set with minimal degree. Van Lambalgen's theorem [24] (see Corollary 6.9.3 in [7]) says that $A \oplus B$ is ML-random if and only if A is ML-random and B are A-ML-random. Thus, X is ML-random relative to Y. In particular, X is not computable. Since $\mathbf{0} <_T X \leq_T Z$, we have $X \equiv_T Z$. Similarly, $Y \equiv_T Z$. Hence, $X \equiv_T Y$, which contradicts the fact that X is ML-random relative to Y. □

A sequence $X \in 2^\omega$ is called *high* if $\emptyset'' \leq_T X'$. If $X \equiv_T Y$ and X is high, then Y is also high because $X' \equiv_T Y'$. A degree $\mathbf{a}$ is called high if $\mathbf{a}$ contains a high sequence, or equivalently if all sequences in $\mathbf{a}$ are high. Highness is an important property in the study of computable randomness. In particular, Nies, Stephan and Terwijn [19] showed that every high degree contains a computably random set.

The last piece is finding a high minimal degree, whose existence is known in computability theory.

Theorem 3.2.

$$\mathrm{CR} <_w \mathrm{MLR}$$

Proof. We begin by showing that CR $\leq_w$ MLR. Every ML-random set is computably random, so every ML-random set computes a computably random set. In fact CR $\leq_s$ MLR. Let Id be the Turing functional such that $\mathrm{Id}(X) = X$ for all $X \in 2^\omega$. Then, Φ^X is computably random for every ML-random sequence X. Thus, CR $\leq_S$ MLR.

Suppose, for a contradiction, that MLR $\leq_w$ CR. Let $\mathbf{a}$ be a high minimal Turing degree, whose existence is shown in [4]. Then, there exists a computably random set $X \in \mathbf{a}$. Thus, there should be $Y \in$ MLR such that $Y \leq_T X$. Since the Turing degree of X is minimal and Y can not be computable, we have $X \equiv_T Y$ and the Turing degree of Y is minimal, which contradicts the fact $Y \in$ MLR. □

Notice that MLR $<_w$ CR implies that MLR and CR are distinct. If MLR = CR, they have the same Muchnik and Medvedev degrees. In contrast, different sets can have the same Muchnik or Medvedev degree as we will see MLR $\equiv_w$ DiffR later. Thus, the fact that MLR and CR are different is not enough to show MLR $<_w$ CR. Some proof ideas are already implicit in proofs of the strict inclusions of the randomness notions. One can see this paper as a list of proofs of the strict inclusions only by degree properties, not by construction of a particular set.

Next, we show WR $<_w$ SR. Van Lambalgen's theorem for Schnorr randomness with the usual Turing relativization does not hold and we need uniform (or truth-table) relativization [15]. Instead, we use the fact that every non-high Schnorr random set is ML-random, which is shown in Nies, Stephan and Terwijn [19]. Then we can make a similar argument by minimal degrees.

Another notion we need is hyperimmunity. A Turing degree $\mathbf{a}$ is *hyperimmune* if $\mathbf{a}$ computes a function $f : \omega \to \omega$ that is not dominated by a computable function. Every hyperimmune contains a Kurtz random set that is not Schnorr random ([7, Corollary 8.11.10]).

Now it suffices to show the existence of a degree that is minimal, non-high, and hyperimmune.

Theorem 3.3.

$$\mathrm{WR} <_w \mathrm{SR}$$

Proof. Since WR $\supseteq$ SR, we have WR $\leq_w$ SR. Assume, for a contradiction, that SR $\leq_w$ WR.

Let $\mathbf{a}$ be a minimal (Turing) degree below $\mathbf{0}'$, whose existence was shown in Sacks [21]. Note that $\mathbf{a}$ is not high because no minimal degree below $\mathbf{0}'$ can be high ([4]). Furthermore, since every nonzero degree $\mathbf{0}'$ is hyperimmune ([14]), $\mathbf{a}$ is hyperimmune. Let $X \in \mathbf{a}$ such that $X \in \mathrm{WR} \setminus \mathrm{SR}$. This is possible because every hyperimmune degree contains a Kurtz random set that is not Schnorr random.

Since SR $\leq_w$ WR, there exists a Schnorr random set $Y \leq_T X \in$ WR. Since $\mathbf{a}$ is minimal and Y can not be computable, $Y \in \mathbf{a}$ and Y is not high. Since Y is non-high Schnorr random, Y is already ML-random, and with minimal degree. This is a contradiction. □

Next we show DiffR $<_w$ W2R. Both notions have characterizations by a degree property in ML-randomness, which is suitable for our purpose. A set X is *difference random* if and only if X is ML-random and X is Turing incomplete [9]. A set X is *weakly 2-random* if and only if X is ML-random and the degree of X forms a minimal pair with $\mathbf{0}'$ [8]. We say that two incomputable sets A, B form a minimal pair if every set $Z \leq_T A, B$ is computable. We again make use of van Lambalgen's theorem for ML-randomness.

Theorem 3.4.

$$\mathrm{DiffR} <_w \mathrm{W2R}$$

Proof. Let Ω be a halting probability. Recall that Ω is ML-random and $\Omega \equiv_T \emptyset'$. Let $\Omega_0 \oplus \Omega_1 = \Omega$.

We claim that Ω_0 is difference random Turing below $\mathbf{0}'$. First of all, by van Lambalgen's theorem, Ω_0 is ML-random. Notice that Ω_0 is incomplete, otherwise $\Omega_1 \leq_T \Omega \equiv_T \emptyset' \leq_T \Omega_0$ and this contradicts the fact that Ω_1 is ML-random relative to Ω_0 by van Lambalgen's theorem. Hence, Ω_0 is difference random. Furthermore, $\Omega_0 \leq_T \Omega \equiv_T \emptyset'$.

Now, suppose that W2R $\leq_w$ DiffR. Then, there exists $X \in$ W2R such that $X \leq_T \Omega_0 \leq_T \emptyset'$, but X cannot be a minimal pair with $\mathbf{0}'$, because $X \leq_T X$ and $X \leq_T \emptyset'$ is not computable and X itself is a counterexample. □

Next, we show W2R $<_w$ 2R. If $\mathbf{a}$ is not hyperimmune, the degree $\mathbf{a}$ is called *hyperimmune-free*. Clearly, if $A \leq_T B$ and B has a hyperimmune-free degree, then A has a hyperimmune-free degree.

Theorem 3.5.

$$\mathrm{W2R} <_w \mathrm{2R}$$

Proof. We use the following facts:

- There exists a ML-random set of hyperimmune-free degree ([7, Theorem 8.1.3]).
- If A has hyperimmune-free degree, then A is Kurtz random if and only if A is weakly 2-random (Yu [7, Theorem 8.11.12]).

- Every 2-random degree is hyperimmune (Kautz [11], see [7, Theorem 8.21.2]).

The above two facts imply that there exists a weakly 2-random set of hyperimmnue-free degree, which can not compute a hyperimmune set, nor a 2-random set. □

Finally, we see the relation with Demuth randomness.

Theorem 3.6.

$$\mathrm{DiffR} <_w \mathrm{DemR} <_w 2\mathrm{R}$$

Furthermore, W2R *and* DemR *are incomparable in Muchnik degrees.*

Proof. There exists a weakly 2-random set X that of hyperimmune-free degree. Any set $Y \leq_T X$ is of hyperimmune-free degree, while no Demuth random set is of hyperimmune-free degree ([7, p.316]). Thus, DemR $\not\leq_w$ W2R.

There exists a Demuth random set Z in Δ^0_2 ([7, Theorem 7.6.3]). Any set $W \leq_T Z$ cannot be a minimal pair with $\mathbf{0}'$. Thus, W2R $\not\leq_w$ DemR.

Thus, W2R and DemR are incomparable in Muchnik degrees, which implies DiffR $<_w$ DemR $<_w$ 2R. □

3.2. **ML-randomness and difference randomness.** We have seen the separations of the adjacent randomness notions in Muchnik degrees. From now on we show the equivalence between MLR and DiffR in Muchnik degrees. The equivalence MLR $\equiv_w$ DiffR means that one can compute a difference random set from every ML-random set. The proof is given by constructing a reduction.

Theorem 3.7.

$$\mathrm{MLR} \equiv_w \mathrm{DiffR}$$

Proof. Let $X = Y \oplus Z$ be a ML-random set.

We claim that at least one of Y or Z should be difference random. Suppose that Y is not difference random. By van Lambalgen's theorem, Y is ML-random, so Y is Turing above $\mathbf{0}'$. Again, by van Lambalgen's theorem, Z is ML-random relative to Y. Thus, Z is $\mathbf{0}'$-ML-random, thus 2-random and difference random.

Note that $Y, Z \leq_T X$. Then X can compute a difference random set, which is Y or Z. □

In this proof, we do not know which of Y or Z is difference random, so the reduction is not uniform. In fact, we can not do this uniformly.

Theorem 3.8.

$$\mathrm{MLR} <_s \mathrm{DiffR}$$

The task is as follows. For every Turing functional Φ, we need to construct a ML-random set X such that $\Phi(X)$ is not difference random.

The key notion in the proof is the push-forward measure. Let (X_1, A_1) and (X_2, A_2) be two measurable spaces where A_1 and A_2 are σ-algebra on X_1 and X_2, respectively. Given a measurable map $f : X_1 \to X_2$ and a measure ν on X_1, the *push-forward* m measure of ν by f is defined to be

$$m(B) = \nu(f^{-1}(B)) \text{ for } B \in A_2.$$

If two measurable functions are equal almost everywhere, then they induce the same push-forward measure.

In the following proof, we consider the push-forward measure μ of the uniform measure λ by the Turing functional Φ. Roughly speaking, if the measure μ is very different from λ, then we can find such X because randomness does not preserve by Φ. If the measure μ is close to λ in some sense, then we can construct such X because some properties do preserve by Φ. This strategy is also used in the proof of SR $<_s$ CR. In this case, more concretely, if the measure μ has an atom, then the atom is a counterexample. If the measure is continuous, then the classes of the random sets with respect to μ and λ are not so different in the sense of Turing degree, so we can find a counterexample again. Given a measure ν on a space X, an element $x \in X$ is an *atom* if $\nu(\{x\}) > 0$. In this case we say that ν has an atom. If the measure does not have an atom, then the measure is called *continuous*.

A measure ν on 2^ω is *computable* if the function $\sigma \mapsto \nu(\sigma)$ is computable, or equivalently $\nu(\sigma)$ is computable uniformly in σ. It is known that every atom for a computable measure is computable [7, Lemma 6.12.7]. Any computable points can not be random in any sense. Moreover, the inverse image of the atom should have a random point because its measure is positive. For two continuous computable measures, their random points have 1-1 correspondence in some sense. In particular, Levin-Kautz theorem ([25, 11] and [7, Theorem 6.12.9(iii)]) says that, if ν is a continuous computable measure and $\mathbf{a} > 0$ then $\mathbf{a}$ contains a λ-ML-random set iff $\mathbf{a}$ contains a ν-ML-random set. Here, λ is the uniform measure.

The last theorem we use in the proof is a no-randomness-from-nothing result, which is a converse of the randomness conservation theorem.

The randomness conservation theorem is the following. Let ν be a computable computable measure on 2^ω. Let $\Phi :\subseteq 2^\omega \to 2^\omega$ be a Turing functional defined ν-almost-everywhere. if $X \in 2^\omega$ is ν-ML-random, then $\Phi(X)$

is ML-random with respect to the push-forward measure of ν by Φ The push-forward measure is computable in this case. For a proof, see [2, Theorem 3.2] etc.

No-randomness-from-nothing result is the following. Let ν be a computable computable measure on 2^ω. Let $\Phi :\subseteq 2^\omega \to 2^\omega$ be a Turing functional defined ν-almost-everywhere. Let μ be the push-forward measure of ν by Φ. If $Y \in 2^\omega$ is μ-ML-random, then there exists a ν-ML-random set X such that $\Phi(X) = Y$. This result is really useful in our proof. We do not need to construct X; we only have to find Y with a property.

Proof of 3.8. Since MLR $\supseteq$ DiffR, we have MLR $\leq_s$ DiffR. Assume, for a contradiction, that DiffR $\leq_s$ MLR. Then, there exists a Turing functional $\Phi :\subseteq 2^\omega \to 2^\omega$ such that $\Phi(X)$ is difference random for every ML-random set X.

Let λ be the fair-coin measure on 2^ω. Let μ be the push-forward measure of λ by Φ, that is, $\mu(B) = \lambda(\Phi^{-1}(B))$ for every Borel set B on 2^ω. The measure μ is computable because the measures $\mu([\sigma])$ are uniformly left-c.e. and $\sum_{\sigma \in 2^n} \mu([\sigma]) = 1$ for every n.

Suppose that μ has an atom, that is, $\mu(\{Y\}) > 0$ for some $Y \in 2^\omega$. Then, $\lambda(\Phi^{-1}(\{Y\})) > 0$. Hence, there exists a ML-random set $X \in \Phi^{-1}(\{Y\})$. Then, $Y = \Phi(X)$ and Y is difference random by the assumption. This contradicts the fact that every atom for a computable measure is computable.

Now suppose that μ is continuous (no atom). Notice that the degree $\mathbf{0}' > \mathbf{0}$ contains a ML-random set Ω. By Levin-Kautz theorem, $\mathbf{0}'$ contains a μ-ML-random set Y. By no-randomness-from-nothing for ML-randomness, there exists a λ-ML-random set X such that $Y = \Phi(X)$. However, $\Phi(X)$ is not difference random because $Y \in \mathbf{0}'$, which is a contradiction. □

3.3. **Schnorr randomness and computable randomness.** Now we turn to the relation between Schnorr randomness and computable randomness. As the relation between ML-randomness and difference randomness, we will see SR $\equiv_w$ CR and SR $<_s$ CR. The proof of the former is not difficult, but the one of the latter needs a little work.

We recall some definitions. A *martingale* is a function $d : 2^{<\omega} \to \mathbb{R}^+$ such that $2d(\sigma) = d(\sigma 0) + d(\sigma 1)$ for every $\sigma \in 2^{<\omega}$. Here, $\mathbb{R}^+$ denotes the set of non-negative reals. A set X is not *computably random* if there exists a computable martingale d such that $\sup_n d(X \upharpoonright n) = \infty$. A set X is not *Schnorr random* if and only if there is a computable martingale d and a computable function f such that $d(A \upharpoonright f(n)) \geq n$ for infinitely many n [10]. Thus, the difference between Schnorr randomness and computable randomness is the rate of divergence for computable martingales.

In the definition of Schnorr randomness, we can replace $d(A \upharpoonright f(n)) \geq n$ with $d(A \upharpoonright f(n)) \geq 2^n$ by making f grow faster. Futhermore, we can assume d has the *saving property*: for each σ, τ we have

$$d(\sigma\tau) \geq d(\sigma) - 2.$$

First we see their equivalence in Muchnik degrees.

Theorem 3.9.

$$\mathrm{SR} \equiv_w \mathrm{CR}$$

Proof. Let $X \in$ SR. If X is high, then there exists a computably random set $Y \equiv_T X$, because every high degree contains a computably random set [19]. If X is not high, X is already ML-random (again by[19]), thus computably random. □

Note that the proof is not uniform again. The reduction can not be uniform as the following theorem indicates. This is the main theorem of this paper.

Theorem 3.10.

$$\mathrm{SR} <_s \mathrm{CR}$$

The goal is to show the following: for each Turing functional $\Phi :\subseteq 2^\omega \to 2^\omega$, there exists a Schnorr random set X such that $\Phi(X)$ is (undefined or) not computably random. If $\Phi = \mathrm{id}$ (the identity map), this is equivalent to saying that there exists a Schnorr random set that is not computably random. In fact, we extend the method of the construction of $X \in \mathrm{SR} \setminus \mathrm{CR}$ in [19].

To give a proof idea, let us recall the "martingale strategy". The terminology of the martingales in theory of algorithmic randomness and probability theory comes from this strategy in a coin flipping game. In this game, the player predicts whether the next coin is a head or a tail. The player bets some money, and the capital increases the same amount of money if the player is correct and the capital decreases the amount if the player is incorrect. The martingale strategy is the following strategy. The player first bets 1 dollar (say) to a head (say again). If incorrect, the player bets the money doubled from the previous bet, again and again. The player will be correct at some turn, say at n-th turn and increase his capital by

$$2^n - (1 + 2 + 2^2 + \cdots + 2^{n-1}) = 1.$$

We call this strategy *martingale strategy.* By repeating this martingale strategy, the player will increase the capital to infinity almost surely.

The strategy sounds good, but does not work in reality. This is because the player needs infinite amount of money in hand. If the initial capital is finite and the capital should be non-negative at any turn, the player can not continue the strategy from some point almost surely.

Suppose we know that a head will appear at least one turn in $\log(n)$ turns for some reason. Then, by making the first bet $\frac{1}{n}$, the lost of money is bounded by

$$\sum_{k=1}^{\log(n)} \frac{2^{k-1}}{n} = \frac{2^{\log(n)} - 1}{n} < 1.$$

The capital will increase $\frac{1}{n}$ in $\log(n)$ turns. By repeating this for $n = 1, 2, \cdots$, the player will increase the capital to infinity.

The construction of $X \in \mathrm{SR} \setminus \mathrm{CR}$ makes use of this strategy as follows. Construct a martingale V that dominates all computable martingales. If V is bounded along a set X, then X should be computably random. If the rate of divergence of V along a set X is slower than any computable function, then X should be Schnorr random. We construct a set X so that V does not increase along X except the positions $\{a_n\}$ where we set $X(a_n) = 0$. The positions are too sparse so no computable martingale succeeds fast enough in Schnorr sense. The numbers of candidates of the positions are small enough for some computable martingale to succeed by iterating the martingale strategy.

In our case, for each almost-everywhere-defined functional Φ, we need to construct a set $X \in \mathrm{SR}$ such that $\Phi(X) \notin \mathrm{CR}$. We force $\Phi(X)(a_n) = 0$ in some positions similarly. The difficulty here is that the measure of the class of such sets X (loosely speaking) may be very small or even 0, and some computable martingale succeeds on X fast enough.

We divide the case into two. Let μ be the push-forward measure of λ by Φ. The first case is that μ is "far from" the fair-coin measure λ and we can construct $X \in \mathrm{SR}$ such that $\Phi(X) \notin \mathrm{CR}$ by another reason. The second case is that μ is "close to" the fair-coin measure and we can apply the method of id case. The two cases are separated by the condition $\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$, which was (essentially) suggested by Laurent Bienvenu.

Let me give examples. Consider the map $\Phi : 2^\omega \to 2^\omega,\ X \mapsto 0X$. In this case, $\mathrm{CR}(\mu) \subsetneq \mathrm{CR}(\lambda)$. Look for a set $Y \in (\mathrm{SR} \setminus \mathrm{CR}) \cap [0]$. Then, the set X such that $Y = 0X$ is in $\mathrm{SR} \setminus \mathrm{CR}$. Consider the map $\Phi : 2^\omega \to 2^\omega,\ X \mapsto 0^\omega$. In this case, $0^\omega \in \mathrm{CR}(\mu) \setminus \mathrm{CR}(\lambda)$ and $\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda)$. Then, there exists a Schnorr random set X such that $\Phi(X) = 0^\omega$.

Proof of Theorem 3.10; $\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda)$ *case.* Suppose for a contradiction that there exists a Turing reduction $\Phi :\subseteq 2^\omega \to 2^\omega$ such that $\Phi(X) \in \mathrm{CR}$ for every $X \in \mathrm{SR}$. Since Φ is defined almost everywhere, the push-forward measure μ is computable.

Case 1: $\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda)$

By $\mathrm{CR}(\mu)$ and $\mathrm{CR}(\lambda)$, we denote the class of sets that are computably random relative to μ and λ respectively. Then, there exists a set $Y \in \mathrm{CR}(\mu) \setminus \mathrm{CR}(\lambda)$. By the no-randomness-from-nothing result for computable randomness by Rute [20], there exists a set $X \in \mathrm{CR}(\lambda)$ such that $\Phi(X) = Y$. Thus, $X \in \mathrm{SR}$ but $\Phi(X) \notin \mathrm{CR}$. ☐

For a proof of the other case, we use some analytical lemmas. Let ν, μ be measures on a measurable space. We say that μ is *absolutely continuous* with respect to ν (denoted by $\mu \ll \nu$) if $\mu(A) = 0$ for every set such that $\nu(A) = 0$. If $\mu \ll \nu$ and $\nu \ll \mu$, then we say that μ and ν are *equivalent*. Radon-Nikodym theorem says that, if $\mu \ll \nu$, then there is a measurable set f on the space to $\mathbb{R}$ such that

$$\mu(A) = \int_A f\ d\nu$$

for any measurable set A. The function f is called the *Radon-Nikodym derivative* and denoted by $\frac{d\mu}{d\nu}$.

Bienvenu and Merkle [1] observed that, for computable measures μ and ν on 2^ω, we have

$$\mathrm{CR}(\mu) = \mathrm{CR}(\nu) \quad \Rightarrow \quad \mathrm{MLR}(\mu) = \mathrm{MLR}(\nu) \quad \Rightarrow \quad \mu \text{ and } \nu \text{ are equivalent} \tag{3}$$

where the former implication follows from a theorem in Muchnik, Semenor, and Uspensky [17, Theorem 9.7]. The theorem by Muchnik, Semenor, and Uspensky actually shows that, for computable measures μ and ν, we have

$$\mathrm{MLR}(\nu) \cap \mathrm{CR}(\mu) \subseteq \mathrm{MLR}(\mu).$$

Thus, if $\mathrm{CR}(\nu) \subseteq \mathrm{CR}(\mu)$, we have

$$\mathrm{MLR}(\nu) = \mathrm{MLR}(\nu) \cap \mathrm{CR}(\nu) \subseteq \mathrm{MLR}(\nu) \cap \mathrm{CR}(\mu) \subseteq \mathrm{MLR}(\mu).$$

The proof of the latter implication of (3) actually showed that, if there exists $\mathcal{X}$ such that $\mu(\mathcal{X}) = 0$ and $\nu(\mathcal{X}) > 0$ for two computable measures μ, ν, then $\mathrm{MLR}(\nu) \not\subseteq \mathrm{MLR}(\mu)$. Hence, we have the following implications.

Lemma 3.11. *Let μ, ν be computable measures. Then, we have*

$$\mathrm{CR}(\nu) \subseteq \mathrm{CR}(\mu) \quad \Rightarrow \quad \mathrm{MLR}(\nu) \subseteq \mathrm{MLR}(\mu) \quad \Rightarrow \quad \nu \ll \mu.$$

The following lemma is the key observation. If $\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$, then $\mu \ll \lambda$. By Radon-Nikodym theorem, there exists a measurable function $f : 2^\omega \to \mathbb{R}^+$ such that $\mu(A) = \int_A f\ d\lambda$ for a Borel set A. By this existence of f, we can find find positions that separates the space roughly half for each in measure; loosely speaking we can find the positions on which the measure is close to the fair-coin measure.

Lemma 3.12. *Let $\Phi :\subseteq 2^\omega \to 2^\omega$ be a Turing functional almost everywhere defined. Let μ be the push-forward measure of λ by Φ where λ is the fair-coin measure on 2^ω. Assume that $\mu \ll \lambda$. Then, for each $\sigma \in 2^{<\omega}$, we have*

$$\lim_{n\to\infty} \lambda\{X \in [\sigma]\ :\ \Phi(X)(n) = 0\} = \frac{1}{2}\lambda(\sigma).$$

Proof. Let μ_σ be the measure defined by

$$\mu_\sigma(\tau) = \lambda(\Phi^{-1}([\tau]) \cap [\sigma]) = \lambda\{X \in [\sigma]\ :\ \Phi(X) \in [\tau]\}.$$

Then,

$$\mu_\sigma \ll \mu \ll \lambda.$$

Let $g = \dfrac{d\mu_\sigma}{d\lambda}$ be a Radon-Nikodym derivative and let

$$g_n(X) = \frac{1}{2^{-n}} \int_{[X\upharpoonright n]} g\ d\lambda.$$

By Lévy's zero-one law, $\{g_n\}$ converges in the L^1-norm, that is,

$$\lim_n \int |g_n - g|\ d\lambda = 0.$$

For $n \in \omega$ and $i = 0, 1$, let

$$D(n, i) = \{Y \in 2^\omega\ :\ \Phi(Y)(n) = i\}.$$

Since g_{n-1} looks only at $X \upharpoonright (n-1)$, so $\int_{D(n,0)} g_{n-1} d\lambda = \int_{D(n,1)} g_{n-1} d\lambda$. Then,

$$\begin{aligned}
\left| \int_{D(n,0)} g_n d\lambda - \int_{D(n,1)} g_n d\lambda \right| &\le \left| \int_{D(n,0)} g_n d\lambda - \int_{D(n,0)} g_{n-1} d\lambda \right| + \left| \int_{D(n,1)} g_{n-1} d\lambda - \int_{D(n,1)} g_n d\lambda \right| \\
&\le \int_{D(n,0)} |g_n - g_{n-1}| d\lambda + \int_{D(n,1)} |g_n - g_{n-1}| d\lambda \\
&= \int |g_n - g_{n-1}| d\lambda
\end{aligned}$$

Notice that

$$\int_{D(n,0)} g_n d\lambda + \int_{D(n,1)} g_n d\lambda = \mu_\sigma(2^\omega) = \lambda(\sigma).$$

Thus,

$$\lim_n \int_{D(n,0)} g_n d\lambda = \frac{1}{2}\lambda(\sigma).$$

Finally notice that

$$\int_{D(n,0)} g_n d\lambda = \int_{D(n,0)} g d\lambda = \mu_\sigma(D(n,0)) = \lambda\{X \in [\sigma]\ :\ \Phi(X)(n) = 0\}.$$

□

The lemma enables us to find, for each σ, a position n such that the measure $\lambda\{X \in [\sigma]\ :\ \Phi(X)(n) = 0\}$ is roughly a half of $\lambda(\sigma)$. So when one looks only at these positions, μ looks like the uniform measure. We define such positions inductively. The positions are just the candidates for forcing. Then, there are many candidates for σ, but still finite. Thus, we consider all candidates for σ and look for n sufficiently large for all σ.

Proof of Theorem 3.10. **Case 2:** $\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$

First, we construct a computable function $g : \omega \to \omega$. The values of g will be the candidates of the positions where we force $\Phi(X)(g(n)) = 0$. The function g depends only on Φ. We will define $g(n)$ inductively on n. Let $G = \{g(n)\ :\ n \in \omega\}$.

Let ϵ be a positive rational number sufficiently small. Let b_n be a computable strictly increasing sequence of rationals such that $0 < b_n < 1$ and $\prod_n b_n > 1 - \epsilon$. The values b_n are close to 1. They control the ratio between $\lambda\{X \in [\sigma]\ :\ \Phi(X)(n) = 0\}$ and $\lambda(\sigma)$. The values b_n also control how many bits are needed to compute $\Phi(X)(n)$ as explained below.

As a warmup, we define $g(0)$ first. By Lemma 3.12, we can compute $n \in \omega$ such that

$$\lambda\{X \in 2^\omega\ :\ \Phi(X)(n) = i\} > \frac{1}{2} b_0$$

for $i = 0, 1$. We wish to define n to be $g(0)$. However, there are many possibilities of X such that $\Phi(X)(n) = 0$. In order to construct X inductively, we would like to restrict possibilities to be finite. Notice that we can enumerate all strings τ such that $\Phi(\tau)(n)$ is defined. Since Φ is almost everywhere defined, the measure of all such strings is 1 for each n. In some fixed order search sufficiently large n and the least s such that

$$\lambda(\bigcup\{[\tau] \ : \ \Phi(\tau)(n)[s] = i\}) > \frac{b_0}{2}$$

for both $i = 0, 1$. Then we define $g(0)$ to be this n. For this s, let $a(0)$ be the maximum of the lengths of the strings in the set

$$\{\tau \ : \ \Phi(\tau)(n)[s] \downarrow\}.$$

We also let

$$T(0) = \{\sigma \in 2^{a(0)} \ : \ \tau \preceq \sigma, \ \Phi(\tau)(n)[s] \downarrow\}.$$

We define $g(k)$, $a(k)$, $T(k)$ inductively on k. For each $\sigma \in T(k-1)$, search $n > g(k-1)$ and s such that

$$\lambda(\bigcup\{[\tau] \ : \ \sigma \prec \tau, \ \Phi(\sigma)(n)[s] = i\}) > \frac{b_k}{2}\lambda(\sigma)$$

for both $i = 0, 1$. For each σ, we can find such s by taking sufficiently large n. Since the set $T(k-1)$ is finite, we can computably find sufficiently large n and the least s such that the above inequality holds for all $\sigma \in T(k-1)$. Then we define $g(k)$ to be this n. For this s, let $a(k)$ be the maximum of the lengths of the strings in the set

$$\bigcup_{\sigma \in T(k-1)} \{\tau \ : \ \sigma \prec \tau, \ \Phi(\tau)(n)[s] \downarrow\}.$$

We also let

$$T(k) = \{\rho \in 2^{a(k)} \ : \ \sigma \prec \tau \preceq \rho, \ \Phi(\tau)(n)[s] \downarrow\}.$$

Note that, for each $\sigma \in T(k-1)$,

$$|\{\rho \in T(k) \ : \ \sigma \preceq \rho\}| > b_k 2^{a(k)-a(k-1)}.$$

Similarly,

$$|\{\rho \in T(k) \ : \ \sigma \preceq \rho, \ \Phi(\rho)(n) = 0\}| > \frac{b_k}{2} 2^{a(k)-a(k-1)}.$$

Now we essentially follow the construction of $X \in \mathrm{SR} \setminus \mathrm{CR}$ in [19]. We define ψ as follows:

$$\psi(e, x) = \begin{cases} \langle e, x, s\rangle + 1 & \text{where } s \text{ is the smallest number such that } \Phi_e(x)[s] \downarrow, \\ \uparrow & \text{if } \Phi_e(x) \uparrow. \end{cases}$$

Here, $\langle i, j\rangle$ is a number coding of the ordered pair and define $\langle i, j, k\rangle = \langle\langle i, j\rangle, k\rangle$. Notice that ψ is one-to-one where defined, $\psi(e, x) > x$, and the relation $n \in \mathrm{rng}\psi$ is decidable. Furthermore, the numbers e, x such that $\psi(e, x) = n$ is computable from n.

We define a computable function $p : \omega \to \omega$ as follows:

$$p(n) = \begin{cases} p(x) + 1 & \text{if } \exists x < n, \exists e < \log p(x) - 1, \psi(e, x) = n, \\ n + 4 & \text{otherwise.} \end{cases}$$

Loosely speaking, the number of candidates of forcing positions at stage n is $\log p(n) - 1$. Notice that $\lim_n p(n) = \infty$.

We define a set H_x as follows:

$$H_x = \{\psi(e, x) \ : \ \psi(e, x) \downarrow, e < \log p(x) - 1\}.$$

The set H_x is the candidates of forcing positions at stage x. We assume Φ_0 is total and H_x is not empty for each $x \in \omega$. Notice that H_x is pairwise disjoint and $|H_x| \leq \log p(x) - 1$.

Let $h(x) = \max(H_x)$. We will force $\Phi(X)(g(h(x))) = 0$. The function h may not be monotone, but h grows faster than any computable order in the following sense. Let $f : \omega \to \omega$ be an increasing unbounded computable function. Let e be such that $\Phi_e(x) = 2^{2^{f(x)}}$. Then, $\psi(e, x) > 2^{2^{f(x)}}$. Since $\lim_x p(x) = \infty$, we have $e < \log p(x) - 1$ for almost all x. For such x, we have $h(\log \log x) > f(x)$. Let $\widehat{h}$ be the reordering of h, that is, the strictly increasing function $\widehat{h}$ such that $\{\widehat{h}(n)\} = \{h(n)\}$. Note that $\widehat{h}$ also dominates all computable functions.

Let $\{M_n\}$ be a non-effective enumeration of $\mathbb{Q}_2$-valued computable martingales with initial capital 1. Let $V = \sum_n 2^{-n-1} M_n$. Then, V is a martingale.

Now, we construct $X \in 2^\omega$. We define $\sigma_{-1} \prec \sigma_0 \prec \cdots \prec \sigma_n \prec X$. Let σ_{-1} to be the empty string and $T(-1)$ to be the set containing only the empty string. Suppose we have chosen $\sigma_{n-1} \in T(n-1)$. The definition of σ_n varies depending on whether there exists k such that $\widehat{h}(k) = n$.

If no k satisfies $\widehat{h}(k)=n$, we would like to define σ_n so that V does not increase rapidly along σ_n after σ_{n-1}. This is possible because the measure of $\bigcup\{[\tau] : \sigma_{n-1}\prec\tau\in T(n)\}$ is close to the measure of $[\sigma_{n-1}]$. We claim that there exists $\tau\in T(n)$ such that $\sigma_{n-1}\preceq\tau$ and

$$V(\tau)\leq\frac{1}{b_n}V(\sigma_{n-1}).$$

Suppose not. Since the possibilities of τ is more than $b_n 2^{a(n)-a(n-1)}$,

$$\sum\{V(\tau) : \tau\in T(n),\sigma_{n-1}\preceq\tau\}>\frac{1}{b_n}V(\sigma_{n-1})\cdot b_n 2^{a(n)-a(n-1)}.$$

In contrast, the fairness condition in the definition of martingales imply

$$V(\sigma_{n-1})=2^{-(a(n)-a(n-1))}\sum\{V(\tau) : \tau\in 2^{a_n},\sigma_{n-1}\preceq\tau\}\geq 2^{-(a(n)-a(n-1))}\sum\{V(\tau) : \tau\in T(n),\ \sigma\preceq\tau\},$$

which is a contradiction. Then, define σ_n to be such τ.

If there exists k such that $\widehat{h}(k)=n$, we need to define σ_n so that $\Phi(\sigma_n)(g(n))=0$. We can show that there exists $\tau\in T(n)$ such that $\sigma_{n-1}\preceq\tau$, $\Phi(\tau)(g(n))=0$, and

$$V(\tau)\leq\frac{2}{b_n}V(\sigma_{n-1})$$

in the same way. Then, define σ_n to be such τ.

We claim that X is Schnorr random. Suppose that there exists a computable martingale M_k and a computable order f such that $M_k(X\upharpoonright f(m))>2^m$ for infinitely many m. Let $c(m)$ be the smallest natural number such that $f(m)\leq a(c(m))$. Since f and a are increasing and computable, c is also computable. By the saving property of V,

$$V(X\upharpoonright a(c(m)))\geq V(X\upharpoonright f(m))-2\geq 2^{-k-1}M_k(X\upharpoonright f(m))-2\geq 2^{m-k-1}-2.$$

In contrast,

$$V(X\upharpoonright a(c(m)))\leq\prod_{n=1}^{c(m)}\frac{1}{b_k}\cdot 2^{|\{k : \widehat{h}(k)\leq c(m)\}|}<\frac{1}{1-\epsilon}2^{|\{k : \widehat{h}(k)\leq c(m)\}|}.$$

This contradicts the fact that $\widehat{h}$ dominates all computable functions.

Finally, we claim that $\Phi(X)$ is not computably random. We construct a computable martingale M that succeeds on X. The initial capital of M is 1. The martingale M uses the martingale strategy S. We describe the strategy of M for each $Z=Z(0)Z(1)Z(2)\cdots\in 2^\omega$. The strategy can be divided into countable stages.

At 0-th stage, the strategy uses the martingale strategy with initial bet $\frac{1}{p(0)}=\frac{1}{4}$ only at $g(H_0)$ until the prediction is correct. Notice that $g(H_0)$ is a computable finite set. A little formal definition by induction on σ is as follows. If $|\sigma|\notin g(H_0)$, then define $M(\sigma 0)=M(\sigma 1)=M(\sigma)$. If $|\sigma|\in g(H_0)$, then define $M(\sigma 0)=M(\sigma)+\frac{2^k}{p(0)}$ and $M(\sigma 1)=M(\sigma)-\frac{2^k}{p(0)}$ with an exception explained below. Here k is the number of positions taking risks so far, that is, $k=|\{n\leq|\sigma| : n\in H_0\}|$. The exception is the case that $M(\sigma)-\frac{2^k}{p(0)}<0$ for some σ. In this case, define $M(\tau)=M(\sigma)$ for all $\tau\succ\sigma$ and 0-th stage never ends. Clearly, M keeps the capital non-negative. The 0-th stage ends at the position $g(n)$, that is, the smallest number such that $n\in H_0$ and $Z(g(n))=0$, or never ends if no such n exists. Note that the position $g(n)$ need not be $g(h(0))$. Since $|H_0|\leq\log p(0)$, by the property of the martingale strategy, the capital $M(Z\upharpoonright g(n))$ at the end of the stage is $1+\frac{1}{4}$. Since $n\in H_0$, $p(n)=p(0)+1=5$. Next stage starts from the next position.

The strategy at m-th stage for $m\geq 1$ as follows. Suppose that $(m-1)$-th stage ends the position $g(n)$ and assume that $p(n)=m+4$. We prove this induction on m later. The strategy uses the martingale strategy with initial bet $\frac{1}{m+4}$ only at $g(H_n)$ until the prediction is correct. Note that each element in $g(H_n)$ is larger than $g(n)$ and all candidates has not come yet. Since $H_n\leq\log p(n)=\log(m+4)$, the capital $M(Z\upharpoonright g(n))$ at the end of the m-th stage is $1+\sum_{i=0}^{m}\frac{1}{i+4}$.

Suppose that the m-th stage ends at the position $g(n')$. We claim that $p(n')=m+5$. This is true for $m=0$. For $m\geq 1$, since $n'\in H_n$, there exists $e<\log p(n)-1$ such that $n'=\psi(e,n)$. By the definition of p, since $n<n'$, we have

$$p(n')=p(n)+1=(m-1+5)+1=m+5.$$

Finally, we prove that M succeeds on the set $\Phi(X)$. By construction of X, we have $\Phi(X)(g(h(x)))=0$ for all x. Thus, for each m, there exists $n\in H_m$ such that $\Phi(X)(g(n))=0$. Hence, each stage ends at some point and the supremum of M along X is infinity. □

3.4. **KL-randomness and ML-randomness.** We give a comment on the relation between Kolmogorov-Loveland randomness (KL-randomness) and ML-randomness. We know that each ML-random set is KL-random but it is a long open question whether the inclusion is strict. Merkle, Miller, Nies, Reimann and Stephan showed that, if $X = Y \oplus Z$ is KL-random, then at least one of Y and Z is ML-random ([13, Theorem 12]). This fact immediately implies that KL $\equiv_w$ MLR where KL is the class of all KL-random sets. We do not know whether KL $\equiv_s$ MLR.

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