Continuity of limit computable functions

Kenshi Miyabe, Meiji University

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Motivation

Calculate

The children learn how to calculate:

- $5 + 7$
- 12×23
- 3^4
- $(x + 1)^2 + 2$

Probably necessary, but make the computers do these.

Continuity

Given x, we compute

$$y = \sin x$$

by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

The terms converge to 0 rapidly enough, so

$$(\forall n \in \mathbb{N})(\exists \delta > 0)|x - a| < \delta \Rightarrow |\sin x - \sin a| < 2^{-n}$$

Only need finite information of x.
Possible because the function is continuous.

Computable function

The theory of computation gives a definition of computable functions.

Theorem 1. *Every computable function is continuous.*

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Discontinuous function

Then, the floor function $[\cdot]$ is not continuous, nor computable:

$$[x] = \max\{n \in \mathbb{N} \ : \ n \leq x\},$$

while (almost) any programming language has this function.
The discontinuous points are only integers and they are negligible on the real line.

Question

Question 2. What are computable functions in reality or almost computable functions?

- How to define the negligibility?
- Are they continuous almost everywhere?

The goal is to define a class of such functions, which turns out to be a formulation of randomized algorithm.

Darboux-Froda’s theorem

Darboux-Froda's theorem

Theorem 3 (Darboux-Froda 1929)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be a non-decreasing function. Then, the set of its discontinuous points is at most most countable.*

This theorem is roughly saying that

> ”Every real function in a class well-behaves at almost every point”.

- We give a proof here to understand the topic.
- Probably the simplest.
- Some computable variants, but the negligibility can increase.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is non-decreasing if

$$(\forall x, y) x \leq y \Rightarrow f(x) \leq f(b).$$

A function f is continuous at $x = z$ if

$$\lim_{x \to z} f(x) = f(z).$$

A set A is countable if there exists a bijection between A and $\mathbb{N}$. The set of real numbers is infinite but not countable.

Jump discontinuity

Definition 4. $z \in \mathbb{R}$ is a point of jump discontinuity of f if the two limits from above and below exist and are distinct.

Lemma 5. *Every discontinuous point of a non-decreasing function is a point of jump discontinuity.*

Proof. f : non-decreasing func., z a point, the limits from above and from below exist:

$$\lim_{x \to z-0} f(x) = \sup\{f(x) \ : \ x < z\}.$$

If they are not distinct, f is continuous at $x = z$. □

Proof

Proof. Let A_n be the set of points z such that the two limits differ more than $\frac{f(b)-f(a)}{n}$. If one could choose $n+1$ points $x_0, \cdots, x_n$ from A_n, then

$$\begin{aligned} f(b) \geq & f(a) + \sum_{k=0}^{n} (f(x_k+0) - f(x_k-0)) \\ \geq & f(a) + \frac{f(b)-f(a)}{n} \cdot (n+1), \end{aligned}$$

contradiction. Then, $|A_n| \leq n$ and the set $\bigcup_{n\geq 1} A_n$ of the discontinuous points is countable. □

Computable version

Any non-decreasing partial computable function has an extension outside the set of computable points, which is countable.
The same is true for Lipschitz functions and functions of bounded variation.
(The proofs are in progress.)
So coding a non-decreasing function seems relatively easy if we ignore a countable set.
The floor function is an example.

Baire one function

Baire 1 function

We want to relax the condition.

Definition 6. Every continuous function is in Baire class 0.
The limit of continuous functions is in Baire class 1.
Inductively, the limit of functions in Baire class $n \in \mathbb{N}$ is in Baire class $n+1$.

Example 7. The floor function is in Baire class 1.
The function $f(x) = \begin{cases} 1 & (x \in \mathbb{Q}) \\ 0 & (x \notin \mathbb{Q}) \end{cases}$ is in Baire class 2.

Derivative

The derivative of a function $f : \mathbb{R} \to \mathbb{R}$ is

$$f'(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}.$$

In particular, the derivative (if exists) is in Baire class 1.

Continuously Differentiable

Theorem 8 (Brucker-Leonard 1966). *The following are equivalent for a set $A \subseteq \mathbb{R}$:*

(i) *A is the set of discontinuous points of a derivative.*

(ii) *A is $\mathbf{\Sigma}^0_2$ and meager.*

A Σ^0_2 set is a union of closed set.
A meager set is roughly a "sparse" set, which is also called a set of first category.
The same is true for Baire 1 functions.

Computable version

Theorem 9 (Kuyper-Terwijn 2014)**.** *The following are equivalent for $x \in [0,1]$:*

(i) *Every derivative of a computable function is continuous at x.*

(ii) *x is 1-generic.*

x is 1-generic if it is not in the boundary of any c.e. open set.

The same is true for limit computable functions (pointed out by Brattka).

Computable approximation

The set of the discontinuous points of a derivative can be measure 1.
Limit computable functions are far from ”computability in reality”.
We need to think about measures.

Differentiability

Measure

A measure of a set is something like the size.
Lebesgue measure denoted by μ is the usual measure:

$$\mu([a, b]) = b - a.$$

Roughly, if you randomly choose from a point x in $[0, 1]$, the probability of $x \in A$ is the measure $\mu(A)$ of A.

Measurable functions

A measurable function is something like a function which we can calculate the area.
If $f : [0, 1] \to \mathbb{R}$ is measurable, then the set

$$f^{-1}((a, b))$$

is measurable.

Differentiability

Theorem 10 (Lebesgue 1904)**.** *Every non-decreasing function* $f : [0,1] \to \mathbb{R}$ *is differentiable almost everywhere.*

”Almost everywhere” means that the set of such points has measure 1, or equivalently outside a measure 0 set.

Differentiation Theorem

Theorem 11 (Lebesgue 1910). *For an L^1-function f,*

$$\lim_{x\in B, |B|\to 0} \frac{1}{\mu(B)} \int_B f \, d\mu = f(x)$$

almost everywhere.

A function f is in L^1 if the integral $\int f \, d\mu$ exists. In particular, f is measurable.
A point with this property is called a Lebesgue point.

Computable reals

Recall that any real x has an approximation by rationals $\{q_n\}$ such that $|q_{n+1} - q_n| \leq 2^{-n}$.
If the sequence $\{q_n\}$ is computable, then the real is called a computable real number.

L^1-computability

Definition 12. A function $f :\subseteq [0,1] \to \mathbb{R}$ is called L^1-computable if there exists a computable sequence $\{f_n\}$ of polynomials with $||f_{n+1} - f_n||_1 \leq 2^{-n}$ such that $f(x) = \lim_n f_n(x)$.

Here, $||g||_1 = \int |g| d\lambda$.
$g(x)$ may be large for some x, but the set of such x is small in measure.
Every L^1-computable function is a limit computable function.

Randomness

Theorem 13 (Pathak-Rojas-Simpson 2014)**.** *The following are equivalent for* $x \in [0, 1]$*.*

- x *is a Lebesgue point for every* L^1*-computable functions.*
- x *is Schnorr random.*

The set of Schnorr random points has measure 1. So, every L^1-computable function well-behaves at almost every point.

Lusin's theorem

Theorem 14. *For a measurable function $f : [0, 1] \to \mathbb{R}$ and for any $\epsilon > 0$, there exist a continuous function g and a compact set K such that $\mu([0, 1] \setminus K) < \epsilon$ and $f(x) = g(x)$ for every $x \in K$.*

In particular, an integrable function is almost continuous.

Almost computable

Theorem 15. *For an* L^1*-computable function* $f :\subseteq [0,1] \to \mathbb{R}$*, there exist a sequence* $\{g_n\}$ *of uniformly computable functions and* $\{K_n\}$ *of uniformly co-c.e. closed sets such that* $\mu([0,1] \setminus K_n) \leq 2^{-n}$ *and* $f(x) = g_n(x)$ *for every* $x \in K_n$*.*

Roughly, f has a computable approximation in arbitrary high probability, so a formulation of randomized algorithm. Recall that Miller-Rabin primality test. By modifying a little, we can define computably measurable functions.

Summary

In mathematics, there are many theorems saying that a function with some property has an approximation by continuous functions. These are useful for proving something.
We considered functions with an approximation by computable functions. These are randomized algorithm.

Future work

Question 16. Is every L^1-computable function continuous at every Schnorr random point?

There is no classical counterpart. We can not talk about continuous points for a measurable function.
Being continuous is a condition stronger than being a Lebesgue point. I conjecture the answer is Yes.

End

Thank you.

明治大学
MEIJI UNIVERSITY