The law of the iterated logarithm

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Introduction

SLLN

Borel (1901) showed the strong law of large numbers.

Theorem 1.
Suppose $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$, *i.i.d.*
Let $S_n = \sum_{i=1}^{n} X_i$.
Then,

$$\frac{S_n}{n} \to 0$$

almost surely.

Actually, he showed the existence of ”absolutely normal numbers”. This was surprising at least at that time.

Theorem 2 (Khintchine 1924). *With the same assumption,*

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \ln \ln n}} = 1$$

Notice that $\sup \frac{S_n}{\sqrt{n}} = \infty$ by the central limit theorem and BC2.

Remark. The spelling of the name varies.

EFKP-LIL

Theorem 3 (Erdös-Feller-Kolmogorov-Petrowsky). *Let* ψ *be a positive increasing function. Let*

$$I(\psi) = \int_1^\infty \frac{\psi(\lambda)}{\lambda} \exp(-\psi(\lambda)^2/2) d\lambda$$

If $I(\psi) < \infty$, *then*

$$S_n < \sqrt{n}\psi$$

for almost all n *almost surely. If* $I(\psi) = \infty$, *then*

$$S_n > \sqrt{n}\psi$$

for infinitely many n *almost surely.*

EFKP-LIL 2

This is also called an integral test.
Erdös proved this in 1942. Feller generalized this in 1946. Kolmogorov in 1937 in Lévy's book claimed this without a proof. Petrowsky in 1935 proved this for Brownian motion.
A good reference of this topic would be the Ph.D. thesis

> ”Integral Tests for Brownian Motion and Some Related Processes” by Keprta (1997)

Recall that the limit of random walks can be seen as Brownian motion. Hence, the same law holds for both.

Proofs 1

Many proofs of SLLN have been given by measure-theoretic and via martingales.
Many books omit the LIL probably because the proof is lengthy.
Any proof of the EFKP-LIL for coin-tossing is hard to read.
A proof of the EFKP-LIL for Brownian motion is more accessible, but it requires much prior knowledge. I recommend Section 5.4 in

> Knight. Essentials of Brownian Motion and Diffusion. AMS, 1981.

The goal

Our (tentative) goal is to give a (simple) proof for EFKP-LIL via martingales (without measure-theoretic arguments).
Why important?
Philosophical view:
We need not assume any prior probability. Thus it can directly translate into the argument in game-theoretic probability. Notice that many results precedes the axioms of probability by Kolmogorov 1933. In fact, the main result in the book is the LIL in a general case. Probability theory may not require measures. In other words, I would like to see probability in a different perspective.

The goal 2

Technical view:
We can argue the computability of strategies. This is essential from the point of view of algorithmic randomness. In other words, we would like to look at complexity of the set of paths that do not hold the claim.

We can argue the rate of divergence of the capitals along the path that do not hold the claim. This is closely related to Hausdorff dimension. This is important when we can not assume probability beforehand.

Results

I will give a proof sketch of the following two results.

(i) Validity of the EFKP-LIL via martingales.
(ii) Sharpness of the usual LIL via martingales.

Both proofs are radically simpler than any known proof.

Validity of the EFKP-LIL

Validity

Theorem 4. *Let ψ be a positive increasing continuous function with*

$$I(\psi) = \int_1^\infty \frac{\psi(\lambda)}{\lambda} \exp(-\psi^2(\lambda)) d\lambda < \infty$$

Then,

$$B_t < \sqrt{t}\psi(t)$$

for all sufficiently large t.

Proof idea

(i) We construct a supermartingale $M_t = h(t, B_t)$.
(ii) We check that if $B_t \geq \sqrt{t}\psi(t)$, then M is a large number that goes to infinity as $t \to \infty$.

Supermartingale

A supermartingale w.r.t. X_t is a stochastic process M_t such that

(i) $E(M_t) < \infty$ for all t,

(ii) $E(M_t | \{X_u \ : \ u \leq s\}) = X_s$ for all $s \leq t$.

Roughly speaking, a supermartingle is a process whose expectation is decreasing.

How to use

If M_t is always positive, by the second condition, we can show

$$P(\sup_t M_t = \infty) = 0.$$

This follows from the famous martingale convergence theorem, but actually this also follows from an earlier result by Ville.
Thus, if one wants to show that some event E occurs almost surely, then it suffices to construct a positive martingale M_t that goes to infinity along outside E.

Proof

Recall that $\exp(\kappa B_t - \frac{1}{2}\kappa^2 t)$ is a martingale for any κ. By integrating this w.r.t. κ,

$$M_t = \int_0^1 \exp(\epsilon B_t - \frac{1}{2}\epsilon^2 t)\pi(\epsilon)d\epsilon$$

is also a martingale for an integrable $\pi(\epsilon)$. Set

$$\pi(\epsilon) = \frac{c(\epsilon)}{\epsilon}\psi(1/\epsilon)\exp(-\frac{\psi(1/\epsilon)^2}{2})$$

where $c(\epsilon) \to \infty$ as $\epsilon \to 0$. If $I(\psi) < \infty$, such c exists.

Proof 2

When $B_t \geq \sqrt{t}\psi(t)$, by restricting between $(1-\frac{1}{\psi})\frac{\psi(t)}{\sqrt{t}}$ and $\frac{\psi(t)}{\sqrt{t}}$,

$$M_t \geq \frac{1}{\sqrt{t}}\exp(\frac{\psi^2}{2})c(\frac{\psi}{\sqrt{t}})\frac{\sqrt{t}}{\psi}\psi(\frac{\sqrt{t}}{\psi})\exp(-\frac{\psi(\frac{\sqrt{t}}{\psi})^2}{2}) \to \infty$$

because

$$\frac{\psi(\sqrt{t}/\psi)}{\psi} \to 1,\ \exp(\frac{\psi^2}{2}) \geq \exp(\frac{\psi(\frac{\sqrt{t}}{\psi})^2}{2})$$

Remark. The essential idea was by Prof. Takemura.

Sharpness of the EFKP-LIL

Proof

Suppose $S_n \leq (1-\delta)\sqrt{2n \ln \ln n}$ a.a.
Let C be large and

$$\kappa_1 = (1-\delta)\sqrt{\frac{2 \ln \ln C}{C}}, \;\; \kappa_2 = (1+\delta^*)\kappa_1, \;\; \kappa_3 = (1+\delta^*)\kappa_2$$

Let

$$K_n = 3\prod_{i=1}^{n}(1+\kappa_2 x_i) - \prod_{i=1}^{n}(1+\kappa_1 x_1) - \prod_{i=1}^{n}(1+\kappa_3 x_i)$$

Proof 2

If $S_C \leq \kappa_1 C$ or $S_C \geq \kappa_3 C$, then $K_n \leq 0$. In other cases,

$$K_n < \ln C$$

by simple calculation.
Let

$$L_n = 1 + \frac{1 - K_n}{\ln C}$$

Then, L_n is a positive martingale. Furthermore, if $S_C \leq \kappa_1 C$ then

$$L_C \geq 1 + \frac{1}{\ln C}$$

Proof 3

Consider M_n that uses K_n for $C^k \leq n < C^{k+1}$. For each round, C in K_n should be $C^{k+1} - C^k$.
If $S_n \leq (1-\delta)\sqrt{2n \ln \ln n}$, then

$$\begin{aligned} & S_{C^{k+1}} - S_{C^k} \\ \leq & (1-\delta)\sqrt{2C^{k+1} \ln \ln C^{k+1}} + (1-\delta)\sqrt{2C^k \ln \ln C^k} \\ \leq & (1-\delta/2)\sqrt{2(C^{k+1} - C^k) \ln \ln (C^{k+1} - C^k)} \end{aligned}$$

Finally,

$$M_{C^k} \geq \prod_{i=1}^{k} \left(1 + \frac{1}{\ln(C^{i+1} - C^i)}\right) \to \infty$$

Future work

1. (i) Does a similar argument prove the EFKP-LIL?
2. (ii) What is the exact bound of the rate of divergence of the martingale?
3. (iii) (Validity) When ψ is computable, can the martingale be computable or lower semicomputable? If not, can we construct a ML-random set along which the EFKP-LIL fails?
4. (iv) (Sharpness) When ψ is computable, is there a computable bound from below of the martingale for each set that does not satisfy the EFKP-LIL? If not, can we construct a Schnorr random set along which the sharpness of the EFKP-LIL fails?

End

Thank you.

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