Erdös-Feller-Kolmogorov-Petrowsky law of the iterated logarithm

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Introduction

SLLN

Borel (1901) showed the strong law of large numbers.

Theorem 1.
Suppose $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$, *i.i.d.*
Let $S_n = \sum_{i=1}^{n} X_i$.
Then,

$$\frac{S_n}{n} \to 0$$

almost surely.

LIL

Theorem 2 (Khintchine 1924). *With the same assumption,*

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \ln \ln n}} = 1$$

Here, $\ln$ means the natural logarithm.
By symmetry, $\liminf = -1$.

EFKP-LIL

Theorem 3 (Erdös-Feller-Kolmogorov-Petrowsky)**.** *Let* ψ *be a positive increasing function. Let*

$$I(\psi) = \int_1^\infty \frac{\psi(\lambda)}{\lambda} \exp(-\psi(\lambda)^2/2) d\lambda$$

If $I(\psi) < \infty$, *then*

$$S_n < \sqrt{n}\psi(n)$$

for almost all n *almost surely. If* $I(\psi) = \infty$, *then*

$$S_n > \sqrt{n}\psi(n)$$

for infinitely many n *almost surely.*

EFKP-LIL 2

This is also called an integral test.
Erdös proved this in 1942. Feller generalized this in 1946. Kolmogorov in 1937 in Lévy's book claimed this without a proof. Petrowsky in 1935 proved this for Brownian motion.
A good reference of this topic would be the Ph.D. thesis

> ”Integral Tests for Brownian Motion and Some Related Processes” by Keprta (1997)

Recall that the limit of random walks can be seen as Brownian motion. Hence, the same law holds for both.

Proofs

Many proofs of SLLN have been given by measure-theoretic and via martingales.

Many books omit the LIL probably because the proof is lengthy.

Any proof of the EFKP-LIL for coin-tossing uses the exstended BC, but hard to read.

A proof of the EFKP-LIL for Brownian motion is more accessible, but it requires much prior knowledge. I recommend Ito and McKean (Section 1.8 and 4.12).

The goal

Which randomness notion is sufficient for the EFKP-LIL to hold, when restricting ψ to be computable?
Schnorr randomness is sufficient for most limit theorems to hold. The proof is just by looking computability of the proofs.
However, in our case, $I(\psi)$ may not be computable in general even if ψ is computable. Furthermore, the speed of the convergence and the divergence seems to be related to the speed of the divergence of the martingales.

Idea

Rewrite the proofs only via martingales.

- Easy to check the computability.
- Easy to analyze the relation of Hausdorff dimension.

Huygens' view to probability is not only philosophically interesting, but also mathematically powerful.

Results

Theorem 4. *Let* ψ *be a positive increasing computable function. Let* X *be computably random. Then, the validity of the EFKP-LIL holds.*
There exists a Schnorr random set X *and* ψ *such that* $I(\psi) < \infty$ *but* $S_n > \sqrt{n}\psi(n)$ *i.o.*
Let X *be Schnorr random. Then, the sharpness of the EFKP-LIL holds.*

The proofs have two parts: martingale part and computability part.

Martingale part

Gambler's ruin problem

A has a points and B has b points where $a, b \in \mathbb{N}$.
At each game, the winner got 1 point from the loser.
The game iterates until one of them lost all points.
What is the probability that the final winner is A?

Let $p(a)$ be the probability. Then,

$$p(a) = \frac{1}{2}p(a-1) + \frac{1}{2}p(a+1)$$

and $p(a+b) = 1$ and $p(0) = 0$. So, p is a martingale and $p(a)$ is the probability in the sense of GTP (or Huygens).

Diffusion process

Any diffusion process has a scale function s.
The hitting probability of b in $[a, b]$ from x can be written as

$$\frac{s(x) - s(a)}{s(b) - s(a)}$$

The scale function of the Brownian motion is the identity function.
The normed scale function is again the probability in the sense of GTP.

Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is the stochastic process and the solution of the SDE:

$$dX(t) = -\alpha dt + \sigma B(t)$$

The solution can be written as the transform of the Brownian motion:

$$X(t) = e^{-t} B(\frac{1}{2}(e^{2t} - 1))$$

for $\alpha = \sigma = 1$. The scale function is written as

$$s(x) = \int^x \exp(\frac{\alpha}{\sigma^2} y^2) dy$$

EFKP-LIL

Let e_i be the i-th hitting time to 0 after hitting 1. Then,

$B(t) > h(t)$ for infinitely often

is roughly the same as

The hit to $h(e_i)$ in $[h(e_i), 0]$ from 1 occurs infinitely often.

With some analysis calculation and with martingales forcing BC, we can construct a martingale whose divergence speed is equivalent to $I(\psi)$.

Computability part

CR and SR

A sequence is computably random if no computable martingale succeeds along it.
A sequence X is not Schnorr random if there exists a computable martingale M and a computable order f such that

$$M(X \upharpoonright n) > f(n)$$

for infinitely many n.
So the difference is the divergence speed.

Sharpness

Let ψ be a positive increasing computable function such that $I(\psi) = \infty$. Then, the function

$$x \mapsto \int_1^x \frac{\psi}{t} \exp(-\psi^2/2) dt$$

is a computable order.

If $S_n < \sqrt{n}\psi(n)$ for all n, the martingale constructed above diverges roughly at the same speed of $I(\psi)$, which is a computable order.

Validity

Let ψ be a positive increasing computable function such that $I(\psi) < \infty$, but the convergence is very slow. Notice that $I(\psi)$ is a left-c.e. real and $I(\psi)$ can be ML-random.

We construct a sequence X such that S_n hits $h(e_i)$ when $I(\psi)$ increases $1/k$. The hitting probability is at least $1/k$ and the martingale grows only k-times. The increase occurs too occasionally and the martingale does not succeed in the sense of Schnorr.

Question: For which ψ, is this construction possible?

Summary

- EFKP-LIL is a rare example that Schnorr randomness is not sufficient to hold.
- Rewriting a proof via martingales is useful to analyze computability and to construct a random sequence.

End

Thank you.

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