A tutorial of game-theoretic probability and algorithmic randomness

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Introduction

Randomness

What is the motivation?

(i) to mathematically formalize probability via the frequency approach (von Mises, Kolmogorov)
(ii) required in universal induction via complexity (Solomonoff)
(iii) as an application of computability theory
(iv) to understand the chaotic behavior in ergodic systems, finance, AI ...

Randomness is the notion lacking in foundation of science!

Setting

2^ω: Cantor space
$\sigma \in 2^{<\omega}$: finite binary strings
$[\sigma] = \{X \in 2^\omega \ : \ \sigma \in X\}$: cylinder sets, base elements of the topology on 2^ω
μ: uniform measure defined by $\mu(\sigma) = 2^{-|\sigma|}$

A real x is left-c.e. if there is a computable increasing sequence $\{a_n\}$ of rationals such that $\lim_n a_n = x$.
A function $f : 2^{<\omega} \to \mathbb{R}$ is lower semicomputable if $f(\sigma)$ is unifromly left-c.e.
A open set $U \subseteq 2^\omega$ ie c.e. if $U = \bigcup_{\sigma \in S} [\sigma]$ for some c.e. set S. The measure of a c.e. open set is left-c.e.

ML-randomness

The sequence 0^ω is not random intuitively because it is in a null set.
Idea: random if it is not in a null set that is effectively constructed

A ML-test is a sequence $\{U_n\}$ of uniformly c.e. open sets with $\mu(U_n) \leq 2^{-n}$.
Notice that $\mu(\bigcap_n U_n) = 0$. This set is a null set effectively constructed in the sense of Martin-Löf.
A sequence is ML-random if it avoids every ML-test, that is, it is not in every such null set,
This notion is defined by Martin-Löf 1966.

Martingale

Idea: random if impossible to increase their capital to infinity with betting along the sequence

A supermartingale is a function $M : 2^{<\omega} \to \mathbb{R}^+$ such that

$$2M(\sigma) \geq M(\sigma 0) + M(\sigma 1)$$

This can be seen as a capital process when one wins the same amount as the betting.

Theorem 1 (Schnorr 1971)**.** *$X \in 2^\omega$ is ML-random iff every c.e. supermartingale does not succeed along X.*

Complexity

How $\sigma \in 2^{<\omega}$ complicated? How to measure the information that σ has in it?
Idea: If complicated, no short program code can produce the string.

$$K(\sigma) = \min\{|\tau| \ : \ U(\tau) = \sigma\}$$

where U is the prefix-free universal Turing machine.

Theorem 2 (Levin-Schnorr 1973). $X \in 2^{\omega}$ *is ML-random iff* $K(X \upharpoonright n) > n - O(1)$.

Natural notion!

Solovay reducibility and Ω

Ω is a concrete example of ML-random sequences.
Halting probability:

$$\Omega_U = \sum_{\sigma \in U} 2^{-|\sigma|}$$

- Many properties do not depend on U.
- Since U is prefix-free and universal, $\Omega < 1$.
- Ω is a left-c.e. real.
- Ω is Turing complete.
- Ω is ML-random.

The proof says Ω stabilizes very slowly.

Solovay reducibility (def)

α, β: left-c.e. reals
Idea: β is more random if β is more difficult to approximate than α.
$\alpha \leq_S \beta$ if there exist a partial computable function $f : \mathbb{Q} \to \mathbb{Q}$ and a constant c such that,

$$q \in \mathbb{Q},\ q < \beta \Rightarrow f(q)\downarrow,\ \alpha - f(q) < c(\beta - q)$$

If one has a good approximation q of β, then one also has a good approximation $f(q)$ of α.

Solovay reducibility

- $\leq_S \Rightarrow \leq_T$
- $\leq_S \Rightarrow \leq_K$

Theorem 3 (Kučera-Slaman, and others)**.** *For a left-c.e. real* α*,*

1-random iff Solovay complete iff $\alpha = \Omega_U$

More relation between computability and randomness has been studied.

What is probability?

The three interpretations

In measure-theoretic probability, probability is the size of plausibility.
This notion has three interpretations:

- the limit of frequency
- the degree of belief
- the physical propensity

Ville's theorem

In game-theoretic probability, we call another object probability.
Recall that effective null sets can be characterized via tests, martingales, complexity.
Its quantitative version is Ville's theorem:

Theorem 4. *For a event $E \subseteq 2^\omega$,*

$$P(E) = \inf\{a \in \mathbb{R} \ : \ \exists M \textbf{\textit{s.t.}} M(\lambda) = a, \sup M(X \upharpoonright n) \geq 1 \textit{ for any } X \in E\}$$

where M is a martingale.

Example

$X_i \in \{\pm 1\}$ for $i = 1, 2, \cdots, 2n+1$ and $S = \sum_{i=1}^{2n+1} X_i$.
Suppose $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$.
Then, $P(S > 0) = \frac{1}{2}$.
The number of possible outcomes is 2^{2n+1} and they are equally likely.
The number of outcomes with $S > 0$ is equal to the one with $S < 0$ by symmetry.

Example (cont.)

There exists a betting strategy such that

(i) the initial capital is $\frac{1}{2}$,
(ii) the final capital is ≥ 1 if $S > 0$,
(iii) the capital is always non-negative.

The conditional probability at each stage corresponds to the capital process of the martingale.
This view of probability goes back to Pascal and Huygens but the impact of Laplace's view is so strong that people seem to ignore their view.
We can make a probability theory without using measures.

Randomness and GTP

Randomness

- Use tests, martingales, complexity
- Separation by priority argument
- The strategy should be simple in the sense of computability

GTP

- Use only martingales.
- Compliance
- The strategy should be simple in the degrees (usually of polynomials)

Different proofs allow us look the same theorem differently.

End

Thank you.