Muchnik degrees and Medvedev degrees of the randomness notions

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Background

Randomness hierarchy

$$\mathrm{KR} \supset \mathrm{SR} \supset \mathrm{CR} \supset \mathrm{MLR} \supset \mathrm{DiffR} \begin{matrix} \supset & \mathrm{W2R} & \supset \\ \supset & \mathrm{DemR} & \supset \end{matrix} \mathrm{2R}$$

We know $\exists X \in \mathrm{SR} \setminus \mathrm{CR}$.

Given a Schnorr random, can we compute a computably random?

Mass problems

Definition 1. Let $P, Q \subseteq 2^\omega$.

P is Muchnik reducible to Q ($P \leq_w Q$) if, for every $f \in Q$, there exists $g \in P$ such that $g \leq_T f$.

P is Medvedev reducible to Q ($P \leq_s Q$) if, there exists a Turing functional Φ such that $\Phi^f \in P$ for every $f \in Q$.

The difference is uniformity.

Main results

Muchnik degrees

$$\mathrm{KR} <_w \mathrm{SR} \equiv_w \mathrm{CR} <_w \mathrm{MLR} \equiv_w \mathrm{DiffR} \begin{array}{ccc} <_w & \mathrm{W2R} & <_w \\ <_w & \mathrm{DemR} & <_w \end{array} \mathrm{2R}$$

Medvedev degrees

$$\mathrm{SR} <_s \mathrm{CR}, \ \mathrm{MLR} <_s \mathrm{DiffR}$$

Proof

$\mathrm{SR} \equiv_w \mathrm{CR}$

Let $X \in \mathrm{SR}$.

If X is not high, X is already in CR.

If X is high, X can compute a real in CR.

Both from Nies, Stephan, and Terwijn (2005).

$\mathrm{SR} <_s \mathrm{CR}$

The goal is

$$\forall\Phi\exists X \in \mathrm{SR}[\Phi(X) \notin \mathrm{CR}].$$

When $\Phi = \mathrm{id}$, this means

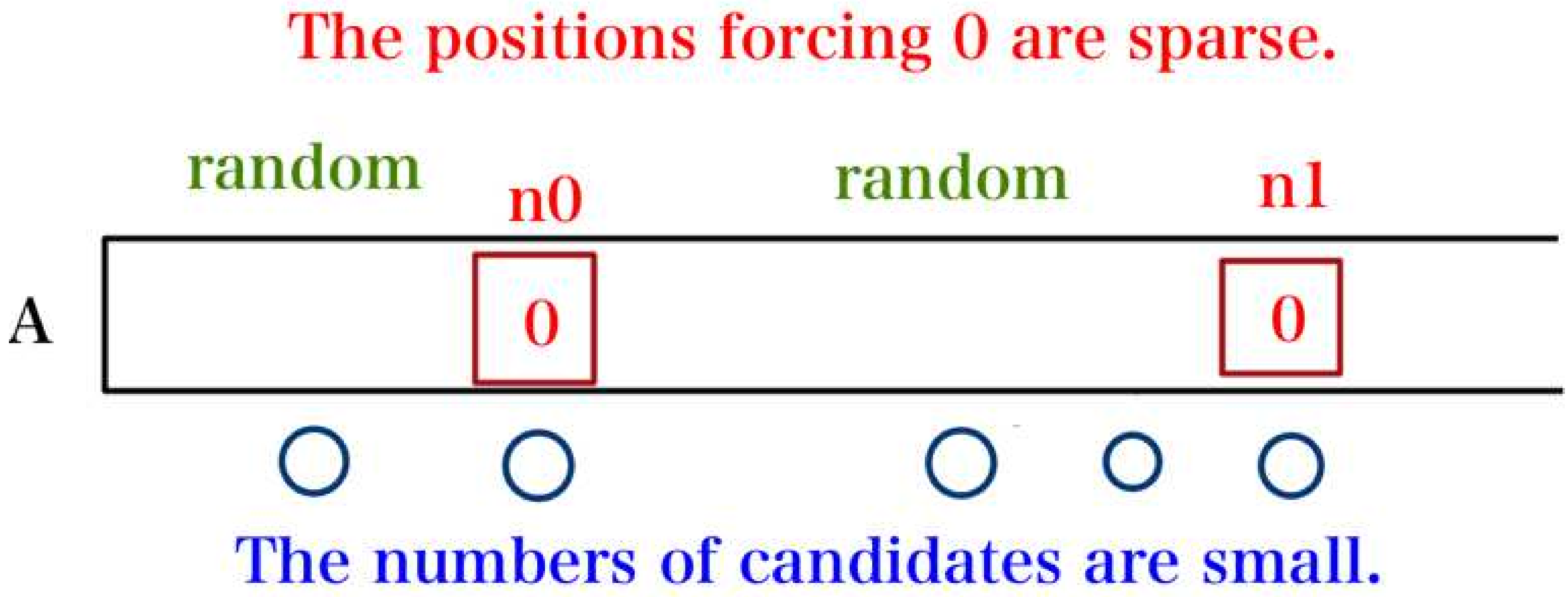
$$X \in \mathrm{SR} \setminus \mathrm{CR}.$$

Thus, we extend the method separating SR and CR.

$\mathrm{SR} \supsetneq \mathrm{CR}$

id case

- Construct a random set A.
- Forcing $A(n_k) = 0$ in sparse positions
  $\Rightarrow$ too sparse not to be Schnorr random
- The number of candidates of n_k is small
  $\Rightarrow$ so small that some computable martingale succeeds on it.


The positions forcing 0 are sparse.
random
n0
random
n1
A
0
0
The numbers of candidates are small.

$\mathrm{SR} <_s \mathrm{CR}$

general case

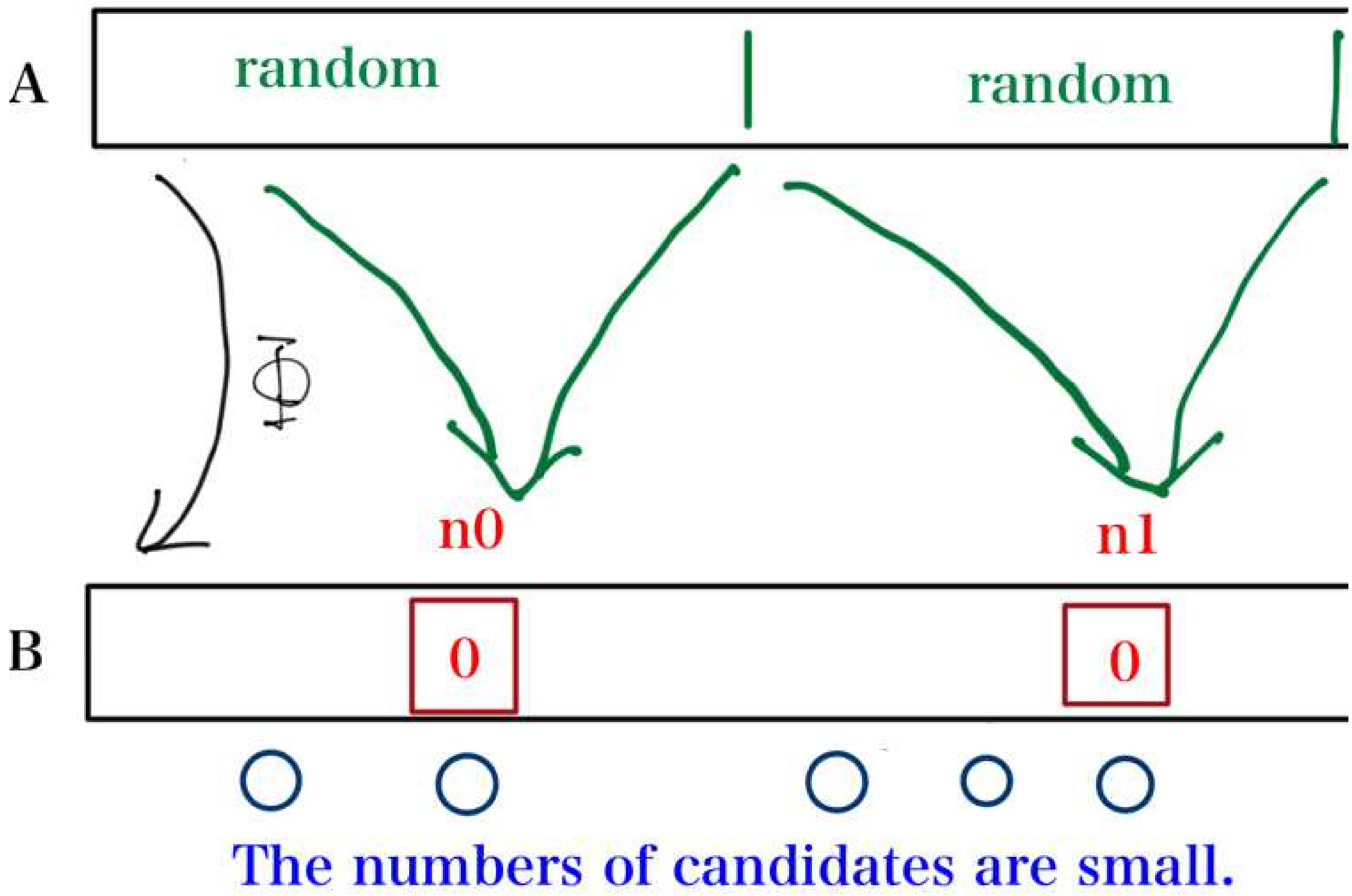
- Construct $A \in \mathrm{SR}$ and $B = \Phi(A) \notin \mathrm{CR}$.
- Forcing $B(n_k) = 0$ in some positions (*)
- The number of candidates of n_k should be small
  $\Rightarrow B \notin \mathrm{CR}$.

The requirement (*) may be strong because

$$\lambda(\{X \in 2^\omega \ : \ \Phi(X)(n_k) = 0\})$$

may be too small (even empty).

$\mathrm{SR} <_s \mathrm{CR}$



$\mathrm{SR} <_s \mathrm{CR}$

We divide the case into two by the induced measure μ.

- μ is ”close to” uniform measure $(\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda))$
  $\Rightarrow$ the same method can be applied
- μ is ”far from” uniform measure $(\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda))$
  $\Rightarrow$ we can show it by a different reason

$\mathrm{CR}(\mu) \not\subseteq \mathrm{CR}(\lambda)$

$$\exists Y \in \mathrm{CR}(\mu) \setminus \mathrm{CR}(\lambda)$$

By no-randomness-from-nothing for CR,

$$\exists X \in \mathrm{CR}\,[\Phi(X) = Y].$$

Then, $X \in \mathrm{SR}$ and $\Phi(X) \notin \mathrm{CR}$.

$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$

Lemma 2 (essentially Bienvenu-Merkle)**.** *Let* μ, ν *be computable measures.*

$$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\nu) \Rightarrow \mathrm{MLR}(\mu) \subseteq \mathrm{MLR}(\nu) \Rightarrow \mu \ll \nu$$

$\ll$ *means absolute continuity.*

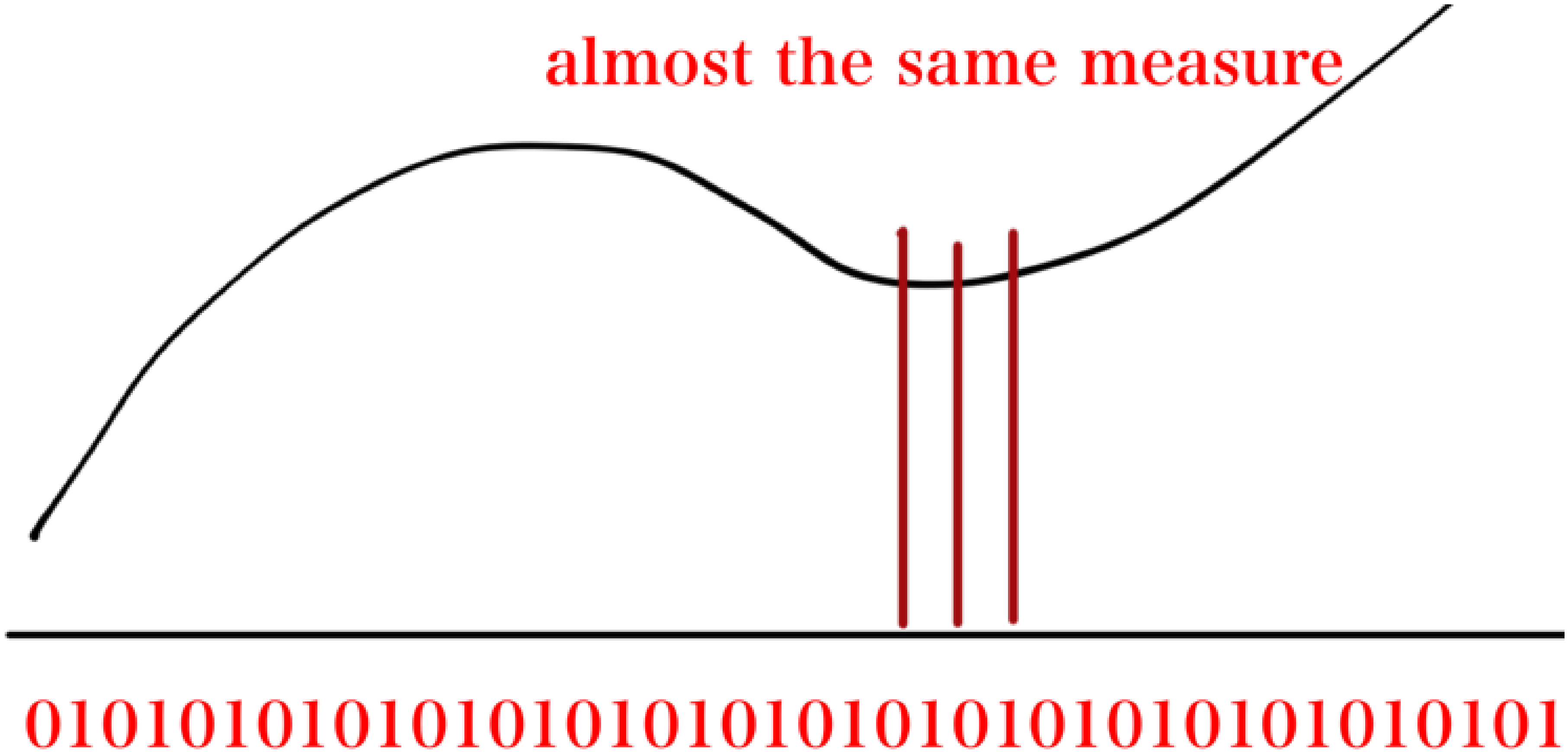
$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$

Lemma 3. *Let Φ be an a.e. computable function. Let μ be the induced measure from Φ and λ. Assume $\lambda \ll \mu$. For each $\sigma \in 2^{<\omega}$,*

$$\lim_{n\to\infty} \lambda\{X \in [\sigma] \ : \ \Phi(X)(n) = 0\} = \frac{1}{2}\lambda(\sigma).$$

Proof. By the Radon-Nikodym theorem and Lévy's zero-one law. □

$\mathrm{CR}(\mu) \subseteq \mathrm{CR}(\lambda)$



Further results

Question

Question 4. Does there exist $A \in \mathrm{SR}$ such that, if $B \leq_{tt} A$ then $B \notin \mathrm{CR}$.
How about wtt?

I conjecture that we can not tt-compute (or wtt-compute) a computably random from a Schnorr random even nonuniformly.

m-degree

Definition 5. $X \leq_m Y$ if $\exists$ comp. f such that

$$n \in X \iff f(n) \in Y$$

Theorem 6. *There exists* $A \in \mathrm{SR}$ *such that, if* $B \leq_m A$ *then* $B \notin \mathrm{CR}$*.*

Every computable subsequence of $A \in \mathrm{SR}$ is not computably random.
So, regularity prevails anywhere!
The proof is a slight extension of the id case.

Summary

- We found two problems that is possible non-uniformly but not possible uniformly.
- Analytical tools are useful to show results in computability. In particular, a.e. computable functions can be studied more from the measure-theoretic perspective.