Muchnik degrees and Medvedev degrees of the randomness notions

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Background

❖ Randomness hierarchy
❖ Mass problems
❖ Main results

Proof

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Background
Randomness hierarchy

KR ⊃ SR ⊃ CR ⊃ MLR ⊃ DiffR ⊃ W2R ⊃ DemR ⊃ 2R

We know ∃X ∈ SR \ CR.

Given a Schnorr random, can we compute a computably random?
Definition 1. Let $P, Q \subseteq 2^\omega$.

$P$ is **Muchnik reducible** to $Q$ ($P \leq_w Q$) if, for every $f \in Q$, there exists $g \in P$ such that $g \leq_T f$.

$P$ is **Medvedev reducible** to $Q$ ($P \leq_s Q$) if, there exists a Turing functional $\Phi$ such that $\Phi^f \in P$ for every $f \in Q$.

The difference is uniformity.
Main results

Muchnik degrees

\[ \text{KR} \prec_w \text{SR} \equiv_w \text{CR} \prec_w \text{MLR} \equiv_w \text{DiffR} \prec_w \text{W2R} \prec_w \text{DemR} \prec_w \text{2R} \]

Medvedev degrees

\[ \text{SR} \prec_s \text{CR}, \quad \text{MLR} \prec_s \text{DiffR} \]
Background

Proof

- $SR \equiv w \ CR$
- $SR <_s CR$
- $SR \supseteq CR$
- $SR \supset CR$
- $SR <_s CR$
- $SR <_s CR$
- $CR(\mu) \not\subseteq CR(\lambda)$
- $CR(\mu) \subseteq CR(\lambda)$
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Further results
Let $X \in SR$.

If $X$ is not high, $X$ is already in $CR$.

If $X$ is high, $X$ can compute a real in $CR$.

Both from Nies, Stephan, and Terwijn (2005).
The goal is

\[ \forall \Phi \exists X \in SR[\Phi(X) \notin CR]. \]

When \( \Phi = \text{id} \), this means

\[ X \in SR \setminus CR. \]

Thus, we extend the method separating \( SR \) and \( CR \).
id case

- Construct a random set $A$.
- Forcing $A(n_k) = 0$ in sparse positions $\Rightarrow$ too sparse not to be Schnorr random
- The number of candidates of $n_k$ is small $\Rightarrow$ so small that some computable martingale succeeds on it.
The positions forcing 0 are sparse.

The numbers of candidates are small.
general case

- Construct $A \in \text{SR}$ and $B = \Phi(A) \not\in \text{CR}$.
- Forcing $B(n_k) = 0$ in some positions (*).
- The number of candidates of $n_k$ should be small
  $\Rightarrow B \not\in \text{CR}$.

The requirement (*) may be strong because

$$\lambda(\{X \in 2^\omega : \Phi(X)(n_k) = 0\})$$

may be too small (even empty).
The numbers of candidates are small.
We divide the case into two by the induced measure $\mu$.

- $\mu$ is "close to" uniform measure ($\text{CR}(\mu) \subseteq \text{CR}(\lambda)$)  
  $\Rightarrow$ the same method can be applied

- $\mu$ is "far from" uniform measure ($\text{CR}(\mu) \not\subseteq \text{CR}(\lambda)$)  
  $\Rightarrow$ we can show it by a different reason
\( \text{CR}(\mu) \not\subseteq \text{CR}(\lambda) \)

By no-randomness-from-nothing for \( \text{CR} \),

\[ \exists X \in \text{CR} \ [\Phi(X) = Y]. \]

Then, \( X \in SR \) and \( \Phi(X) \not\in \text{CR} \).
**Lemma 2** (essentially Bienvenu-Merkle). *Let $\mu, \nu$ be computable measures.*

$$\text{CR}(\mu) \subseteq \text{CR}(\nu) \Rightarrow \text{MLR}(\mu) \subseteq \text{MLR}(\nu) \Rightarrow \mu \ll \nu$$

$\ll$ means *absolute continuity.*
Lemma 3. Let $\Phi$ be an a.e. computable function. Let $\mu$ be the induced measure from $\Phi$ and $\lambda$. Assume $\lambda \ll \mu$. For each $\sigma \in 2^{<\omega}$,

$$\lim_{n \to \infty} \lambda\{X \in [\sigma] : \Phi(X)(n) = 0\} = \frac{1}{2} \lambda(\sigma).$$

Proof. By the Radon-Nikodym theorem and Lévy’s zero-one law.
$\text{CR}(\mu) \subseteq \text{CR}(\lambda)$

almost the same measure

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Further results
Question 4. Does there exist $A \in SR$ such that, if $B \leq_{tt} A$ then $B \notin CR$.

How about wtt?

I conjecture that we can not tt-compute (or wtt-compute) a computably random from a Schnorr random even nonuniformly.
Definition 5. $X \leq_m Y$ if $\exists$ comp. $f$ such that

$$n \in X \iff f(n) \in Y$$

Theorem 6. *There exists* $A \in SR$ *such that, if* $B \leq_m A$

*then* $B \not\in CR$.

Every computable subsequence of $A \in SR$ is not computably random.
So, regularity prevails anywhere!
The proof is a slight extension of the id case.
Summary

- We found two problems that is possible non-uniformly but not possible uniformly.
- Analytical tools are useful to show results in computability. In particular, a.e. computable functions can be studied more from the measure-theoretic perspective.