Schnorr triviality via decidable machines

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Background

Triviality

The following are equivalent for a real $A \in 2^\omega$:

(i) A is K-trivial.
(ii) A is low for ML-randomness.
(iii) A is low for K.
(iv) A is a base for ML-randomness.

The following are equivalent for a real $B \in 2^\omega$:

(i) B is Schnorr trivial.
(ii) B is uniformly low for Schnorr randomness.
(iii) B is uniformly low for computable measure machines.
(iv) B is a base for uniform Schnorr tests.

Triviality - comment

We have a Schnorr-randomness version of each notion.

Obtaining these result was far from by straightforward modification.
In fact, many researchers introduced many notions, most of which are not equivalent to Schnorr triviality.

Decidable machines

ML-random	prefix-free machine
Schnorr random	computable measure machine prefix-free decidable machine

Table 1: prefix-free case

ML-random	plain machine
Schnorr random	total machine

Table 2: plain case

Goal

No characterization of K-triviality via plain machines.
Some characterizations of Schnorr triviality via prefix-free decidable machines and total machines.

Study Schnorr triviality and lowness via decidable machines,
of which we do not have straightforward counterparts in ML-random case.
Hopefully, any suggestion to the study of C.

Lowness

ML-Randomness

The following are equivalent for $X \in 2^\omega$:

(i) X is ML-random.

(ii) $K(X \upharpoonright n) > n - O(1)$ (Levin-Schnorr, Chaitin 1970s)

(iii) $C(X \upharpoonright n) > n - K(n) - O(1)$ (Miller-Yu 2008)

where K is the prefix-free Kolmogorov complexity and C is the plain Kolmogorov complexity.

Schnorr Randomness

The following are equivalent for $X \in 2^\omega$:

(i) X is Schnorr random

(ii) $K_M(X \upharpoonright n) > n - O(1)$ for every computable measure machines M (Downey-Griffiths 2004)

(iii) $K_M(X \upharpoonright n) > n - f(n) - O(1)$ for every prefix-free decidable machine M and every computable order f (Bienvenu-Merkle 2007)

(iv) $C_M(X \upharpoonright n) > n - K_N(n) - O(1)$ for every total machine M and every computable measure machine N (Miyabe 2016)

Decidable machines

An order is a computable function $f : \omega \to \omega$ that is unbounded and nondecreasing.

A machine is called decidable if its domain is computable.
The measure of a machine $M :\subseteq 2^{<\omega} \to 2^{<\omega}$ is

$$\sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|},$$

which is left-c.e. but not computable in general.
A computable measure machine is a machine whose measure is computable.
Every computable measure machine is decidable.

Lowness

$A \in 2^\omega$ is low for K if $K(\sigma) \leq K^A(\sigma) + O(1)$.
Lowness for K is equivalent to K-triviality.

$A \in 2^\omega$ is uniformly low for computable measure machines if $\forall M$: u.c.m.m. $\exists N$: c.m.m. s.t.

$$K_N(\sigma) \leq K_{M^A}(\sigma) + O(1).$$

This is equivalent to comp. tt-traceability (Miyabe 2011), which in turn is equivalent to Schnorr triviality (Franklin-Stephan 2010).

Lowness via pdm, tm

Theorem 1 (M.)**.**
The following are equivalent for $A \in 2^\omega$:

(i) A *is Schnorr trivial.*

(ii) $\forall M$: *updm* $\forall f$: *order* $\exists N$: *pdm s.t.*

$$K_N(n) \leq K_{M^A}(n) + f(n).$$

(iii) $\forall M$: *utm* $\forall f$: *order* $\exists N$: *tm s.t.*

$$C_N(n) \leq K_{M^A}(n) + f(n).$$

Reducibility version

Recall that

$$\leq_{LK} \iff \leq_{LR},$$

which is a reducibility version of the equivalence between lowness for K and lowness for MLR.

The equivalence above also has a corresponding reducibility version.

Another remark

The results above were inspired by the following result:

Theorem 2 (Bienvenu-Merkle 2007)**.** A *is computably traceable iff*
$\forall M$: *pdm with oracles* $\forall h$: *order* $\exists N$: *pdm s.t.*

$$K_N(\sigma) \leq K_M^A(\sigma) + h(K_M^A(\sigma)) + O(1).$$

Computable traceability is equivalent to Turing lowness for Schnorr randomness.
The complexities w.r.t. a uniform machine can be computably bounded from below.

Triviality

Triviality

$A \leq_K B$ if

$$K(A \upharpoonright n) \leq K(B \upharpoonright n) + O(1).$$

K-trivial reals are the bottom class in K-reducibility.

$A \leq_{Sch} B$ if $\forall M$: c.m.m. $\exists N$: c.m.m. s.t.

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + O(1).$$

Schnorr trivial reals are the bottom class in Schnorr reducibility.

Via decidable machines

Theorem 3 (M. 2015)**.**
$A \leq_{Sch} B$ *iff*
$\forall M$: *pdm* $\forall f$: *order* $\exists N$: *pdm s.t.*

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + f(n) + O(1).$$

In particular,

$$\leq_{dm} \Rightarrow \leq_{Sch} .$$

The converse (probably) does not hold.

Via total machines

The following is from Hölzl-Merkle 2010.
A set A is totally i.o. complex if $\exists g$: order s.t.
$\forall M$: tm $\exists^\infty n \in \omega$

$$C_M(A \upharpoonright g(n)) \geq n.$$

They showed that its negation is equivalent to computable tt-traceability, which in turn is equivalent Schnorr triviality.
So Schnorr triviality can be characterized via total machines!!

Question

Note that the negation is equivalent to
$\forall N$: tm $\forall f$: order $\exists N$: tm s.t.

$$C_N(A \upharpoonright n) \leq C_M(n) + f(n).$$

Question 4. $A \leq_{Sch} B$ iff
$\forall N$: tm $\forall f$: order $\exists N$: tm s.t.

$$C_N(A \upharpoonright n) \leq C_M(B \upharpoonright n) + f(n)?$$

I have a proof sketch (with calculation) of ”if” direction,
I conjecture ”only if” direction does not hold.
Any suggestion to $\leq_C \Rightarrow \leq_K$.

Question 2

Question 5. We have characterizations of ML-randomness via decidable machines and total machines.
Can we say anything about K-triviality via decidable machines and total machines?

Proof

Schnorr reducibility

Theorem 6 (M. 2015 again)**.**

$$\leq_{Sch} \iff \leq_{wdm},$$

which means that the following are equivalent for $A, B \in 2^\omega$*:*

(i) $\forall M$: *c.m.m.* $\exists N$: *c.m.m. s.t.*

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + O(1).$$

(ii) $\forall M$: *pdm* $\forall f$: *order* $\exists N$: *pdm s.t.*

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + f(n) + O(1).$$

Key observation

prefix-free machine

$\exists U$: prefix-free s.t. $\forall V$: prefix-free

$$K_U(n) \le K_V(n) + O(1).$$

U is called universal or optimal.

Key observation 2

prefix-free decidable machine
$\exists M$: pdm s.t. $\forall N$: prefix-free

$$K_M(n) \leq K_N(n) + O(1)$$

for infinitely many n. So M is i.o. optimal.

The pdm is essentially the same as time-bounded Kolmogorov complexity $K^t(\sigma)$.
The same holds for a function $t = O(|\sigma|^2)$.

Key observation 3

computable measure machine
$\forall M$: c.m.m. $\exists N$: c.m.m. $\exists f$: order s.t.

$$K_N(n) + f(n) \leq K_M(n).$$

So no c.m.m. is optimal even in the sense of i.o.
Notice that f can be taken as a computable function.

Proof 1

Lemma 7. $\forall M$: *pdm* $\forall g$: *order* $\exists N$: *c.m.m. s.t.*

$$K_N(\sigma) \leq K_M(\sigma) + g(|\sigma|) + O(1).$$

This implies $\leq_{Sch} \Rightarrow \leq_{wdm}$.
Suppose $A \leq_{Sch} B$. M : pdm, g : order.
$\exists M'$: c.m.m. s.t.

$$K_{M'}(B \upharpoonright n) \leq K_M(B \upharpoonright n) + g(n) + O(1).$$

By $A \leq_{Sch} B$, $\exists N$: c.m.m. s.t.

$$K_N(A \upharpoonright n) \leq K_{M'}(B \upharpoonright n) + O(1).$$

Combine these and notice that N is a pdm.

Proof 2

Lemma 8. $\forall M$: *c.m.m* $\exists N$: *pdm* $\exists g$: *order s.t.*

$$K_N(\sigma) + g(|\sigma|) \leq K_M(\sigma) + O(1).$$

This implies $\leq_{wdm} \Rightarrow \leq_{Sch}$.
Suppose $A \leq_{wdm}$. M : c.m.m..
$\exists M'$: pdm $\exists f$: order s.t.

$$K_{M'}(B \upharpoonright n) \leq K_M(B \upharpoonright n) - g(n) + O(1).$$

Since $A \leq_{wdm} B$, $\exists N'$: pdm s.t.

$$K_{N'}(A \upharpoonright n) \leq K_{M'}(B \upharpoonright n) + g(n)/2 + O(1).$$

Proof 3

$\exists N$: c.m.m s.t.

$$K_N(A \upharpoonright n) \leq K_{N'}(A \upharpoonright n) + g(n)/2 + O(1).$$

By combining these, we have

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + O(1).$$

Summary

- Schnorr triviality can be characterized
  - via complexity and lowness
  - w.r.t. computable measure machines, prefix-free decidable machines, total machines.

  It has many characterizations and is really a robust notion.
- The situation seems very different from K-triviality. Is there any suggestion to that?

Thank you for listening.