Uniform relativization

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19 July 2019

CiE2019@Durham, UK

Background

Lowness

The jump of $A \subseteq \omega$ is the halting problem relative to A:

$$A' = \{n \ : \ \Phi_n^A(n)\downarrow\}$$

For any A, B,

$$A \leq_T B \ \Rightarrow \ A' \leq_T B'$$

A is low if

$$A' \equiv_T \emptyset'$$

which is close to computable w.r.t. the jump operator.

Low for ML-randomness

For any $A, B \subseteq \omega$,

$$A \leq_T B \Rightarrow \mathrm{MLR}(B) \subseteq \mathrm{MLR}(A)$$

A is low for ML-randomness if

$$\mathrm{MLR}(A) = \mathrm{MLR}$$

Surprisingly, it has many characterizations such as

(i) lowness for K,
(ii) K-triviality,
(iii) being a base for ML-randomness.

Uniformly Low for SR

For Schnorr randomness, we have the following equivalence:

(i) uniformly low for Schnorr randomness,
(ii) uniformly low for computable measure machines,
(iii) Schnorr triviality,
(iv) being a base for uniform Schnorr tests.

We need uniform relativization.

Goal

For the former half,
we give a basic introduction to uniform relativization.
(This finding was almost a decade ago.)

For the latter half,
we give further equivalences via prefix-free decidable machines and total machines,
no counterpart in ML-randomness.

Uniform relativization

Turing and tt-reducibility

A is Turing reducible to B if A is computable by a Turing machine with an oracle B.
In this case, it is often called that A is computable (Turing) relative to B.

A is truth-table reducible to (or tt-reducible to) B if the reduction is total, which means the computation works for any oracle $X \subseteq \omega$.
Some researchers say that A is tt-computable relative to B.
Roughly speaking, uniform relativization is this tt-version of relativization.

Open set

Cantor space 2^ω is the class of infinite binary sequences equipped with the topology generated by the cylinder sets

$$[\sigma] = \{X \in 2^\omega \ : \ \sigma \prec X\}$$

as a basis.

An open set U on 2^ω is called c.e. if there exists a computable sequence $\{\sigma_n\}$ of finite binary strings such that

$$U = \bigcup_n [\sigma_n]$$

This is sometimes called inner approximation.

Representation

An open set U is c.e. Turing relative to $A \in 2^\omega$ if there exists a sequence $\{\sigma_n\}$ computable relative to A such that $U = \bigcup_n [\sigma_n]$.
Let $\mathcal{O}$ be the class of open sets. Define $\theta : \omega^\omega \to \mathcal{O}$ as follows: For an input $p = \{p(n)\}_n \in \omega^\omega$,

$$\theta(p) = \bigcup_n [\sigma_{p(n)}]$$

where σ_k is the k-th binary string.
Then, U is c.e. Turing relative to A iff there exists a computable function $\Phi :\subseteq 2^\omega \to \omega^\omega$ such that $U = \theta(\Phi(A))$ as usual in computabele analysis.

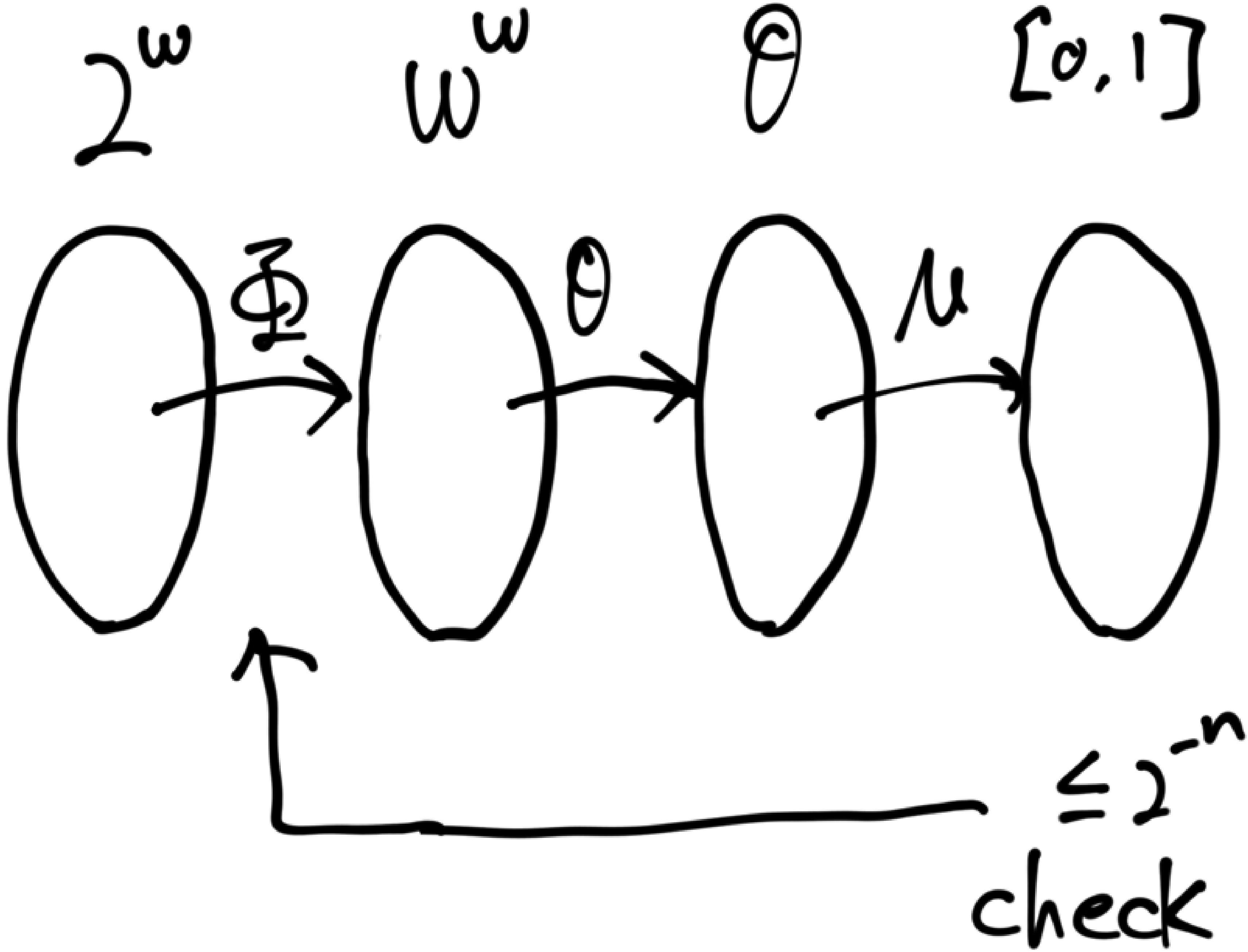
Uniform relativization

U is c.e. Turing relative to A iff $\exists \Phi :\subseteq 2^\omega \to \omega^\omega$ (partial comp.) s.t. $U = \theta(\Phi(A))$.
U is c.e. uniformly relative to A iff $\exists \Phi : 2^\omega \to \omega^\omega$ (total comp.) s.t. $U = \theta(\Phi(A))$.

For a given partial Φ, we can construct a total computable function $\hat{\Phi}$ extending Φ (by adding the special string indicating the empty set on 2^ω).
Thus, U is c.e. Turing relative to A if and only if U is c.e. uniformly relative to A.

With measure restriction

Suppose that U is c.e. Turing relative to A with measure $\leq 2^{-n}$ via a partial comp. Φ.
Then, we can construct a total comp. $\hat{\Phi}$ extending Φ by enumerating the strings as long as the measure is $\leq 2^{-n}$.
Again, Turing relativization is equivalent to uniform relativization in this case.


2^ω
ω^ω
Θ
$[0,1]$
Φ
θ
μ
$\leq 2^{-n}$
check

With computable measure

Finally suppose that U is c.e. Turing relative to A with A-comp. measure $\leq 2^{-n}$ via

(i) a partial comp. Φ (to compute U) and

(ii) another partial comp. $f : 2^\omega \to \mathbb{R}$ (to compute the measure).

Then, we can not extend them to total comp.
This can be proved by using the difference between Turing reducibility and tt-reducibility.

Randomness

A Martin-Löf test (ML-test) is a comp. seq. $\{U_n\}$ of c.e. open sets with measure $\leq 2^{-n}$. A set X is ML-random if $X \notin \bigcap_n U_n$ for any ML-test.
This can be relativized, but Turing and uniformly relativized ML-randomness are the same.

A Schnorr test is a ML-test with computable measures.
Schnorr randomness is defined similarly.
It turns out that there exists a set A such that Schnorr randomness Turing relative to A is different from Schnorr randomness uniformly relative to A.
Ques. For which one? high sets?

Other characterizations

Randomness can be characterized via complexity, martingales, integral tests.
Randomness with uniform relativization also can be done via them.

tt vs uniform

For a random set, there is no reduction to the oracle, so this is not about the reduction but about the relativization.

Why not calling tt-relativization?
The oracle space may be $[0, 1]$; in that case the relativization depends on the names.
We need to consider $\mathcal{O}$ and $[0, 1]$, not appropriate to call them tt-reduction.
The key is the totality of the functions, and we require the functions work for all oracles uniformly.

Lowness

ML-Randomness

The following are equivalent for $X \in 2^\omega$:

(i) X is ML-random.

(ii) $K(X \upharpoonright n) > n - O(1)$ (Levin-Schnorr, Chaitin 1970s)

(iii) $C(X \upharpoonright n) > n - K(n) - O(1)$ (Miller-Yu 2008)

where K is the prefix-free Kolmogorov complexity and C is the plain Kolmogorov complexity.

Schnorr Randomness

The following are equivalent for $X \in 2^\omega$:

(i) X is Schnorr random

(ii) $K_M(X \upharpoonright n) > n - O(1)$ for every computable measure machines M (Downey-Griffiths 2004)

(iii) $K_M(X \upharpoonright n) > n - f(n) - O(1)$ for every prefix-free decidable machine M and every computable order f (Bienvenu-Merkle 2007)

(iv) $C_M(X \upharpoonright n) > n - K_N(n) - O(1)$ for every total machine M and every computable measure machine N (Miyabe 2016)

Decidable machines

An order is a computable function $f : \omega \to \omega$ that is unbounded and nondecreasing.

A machine is called decidable if its domain is computable.
The measure of a machine $M :\subseteq 2^{<\omega} \to 2^{<\omega}$ is

$$\sum_{\sigma \in \mathrm{dom}(M)} 2^{-|\sigma|},$$

which is left-c.e. but not computable in general.
A computable measure machine is a machine whose measure is computable.
Every computable measure machine is decidable.

Lowness

$A \subseteq \omega$ is low for MLR if $\mathrm{MLR}(A) = \mathrm{MLR}$.
A is low for K if $K(\sigma) \leq K^A(\sigma) + O(1)$.
They are equivalent!

A is unif. low for SR if $\mathrm{SR}^*(A) = \mathrm{SR}$.
A is uniformly low for computable measure machines if $\forall M$: u.c.m.m. $\exists N$: c.m.m. s.t.

$$K_N(\sigma) \leq K_{M^A}(\sigma) + O(1).$$

They are equivalent (Miyabe 2011, Franklin-Stephan 2010).

Lowness via pdm, tm

Theorem 1 (M.).
The following are equivalent for $A \in 2^\omega$:

(i) A *is unif. low for Schnorr randomness*

(ii) $\forall M$: *updm* $\forall f$: *order* $\exists N$: *pdm s.t.*

$$K_N(n) \leq K_{M^A}(n) + f(n).$$

(iii) $\forall M$: *utm* $\forall f$: *order* $\exists N$: *tm s.t.*

$$C_N(n) \leq K_{M^A}(n) + f(n).$$

No characterization of lowness for MLR via C is known.

Reducibility version

Recall that

$$\leq_{LK} \iff \leq_{LR},$$

which is a reducibility version of the equivalence between lowness for K and lowness for MLR.

The equivalence above also has a corresponding reducibility version.

Another remark

The results above were inspired by the following result:

Theorem 2 (Bienvenu-Merkle 2007)**.** A *is computably traceable iff*
$\forall M$: *pdm with oracles* $\forall h$: *order* $\exists N$: *pdm s.t.*

$$K_N(\sigma) \leq K_M^A(\sigma) + h(K_M^A(\sigma)) + O(1).$$

Computable traceability is equivalent to Turing lowness for Schnorr randomness.
The complexities w.r.t. a uniform machine can be computably bounded from below.

Triviality

Triviality

$A \leq_K B$ if

$$K(A \upharpoonright n) \leq K(B \upharpoonright n) + O(1).$$

K-trivial reals are the bottom class in K-reducibility.

$A \leq_{Sch} B$ if $\forall M$: c.m.m. $\exists N$: c.m.m. s.t.

$$K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + O(1).$$

Schnorr trivial reals are the bottom class in Schnorr reducibility.

Via decidable machines

Theorem 3 (M. 2015)**.**
$A \leq_{Sch} B$ *iff*
$\forall M$: *pdm* $\forall f$: *order* $\exists N$: *pdm s.t.*

$$K_N(A \restriction n) \leq K_M(B \restriction n) + f(n) + O(1).$$

In particular, Schnorr triviality can be characterized via pdm.

Via total machines

The following is from Hölzl-Merkle 2010.
A set A is totally i.o. complex if $\exists g$: order s.t.
$\forall M$: tm $\exists^\infty n \in \omega$

$$C_M(A \upharpoonright g(n)) \geq n.$$

They showed that its negation is equivalent to computable tt-traceability, which in turn is equivalent Schnorr triviality.
So Schnorr triviality can be characterized via total machines!!

Question

Note that the negation is equivalent to
$\forall M$: tm $\forall f$: order $\exists N$: tm s.t.

$$C_N(A \upharpoonright n) \leq C_M(n) + f(n).$$

Question 4. $A \leq_{Sch} B$ iff
$\forall M$: tm $\forall f$: order $\exists N$: tm s.t.

$$C_N(A \upharpoonright n) \leq C_M(B \upharpoonright n) + f(n)?$$

Any suggestion to $\leq_C \Rightarrow \leq_K$.

	U-Low	U-Low red.	Triviality	Triviality red.
Random	Def	Def	-	-
c.m.m.	Yes	Yes	Yes	Def
p.d.m.	Yes	Yes	Yes	Yes
t.m.	Yes	Yes	Yes	?

End

Thank you for listening.