Uniform relativization

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Abstract. This paper is a tutorial on uniform relativization. The usual relativization considers computation using an oracle, and the computation may not work for other oracles, which is similar to Turing reduction. The uniform relativization also considers computation using oracles, however, the computation should work for all oracles, which is similar to truth-table reduction. The distinction between these relativizations is important when we relativize randomness notions in algorithmic randomness, especially Schnorr randomness. For Martin-Löf randomness, its usual relativization and uniform relativization are the same so we do not need to care about this uniform relativization.
We focus on two specific examples of uniform relativization: van Lambalgen's theorem and lowness. Van Lambalgen's theorem holds for Schnorr randomness with the uniform relativization, but not with the usual relativization. Schnorr triviality is equivalent to lowness for Schnorr randomness with the uniform relativization, but not with the usual relativization. We also discuss some related known results.

Keywords: uniform relativization · Schnorr randomness · van Lambalgen's theorem · lowness

1 Introduction

1.1 Relativization

In computability theory, many notions are relativized via oracle Turing machines. As an example, a set $A \subseteq \mathbb{N}$ is called *computable* if it is computable by a Turing machine. An oracle Turing machine is a Turing machine with an oracle tape, which is a one-way infinite tape. If one uses $B \subseteq \mathbb{N}$ as an oracle, the oracle Turing machine can ask whether $k \in B$ during the computation. If A is computable by a Turing machine with an oracle B, then we say that A is *Turing reducible to* B or that A is *computable relative to* B. Similarly, many notions, results, and proofs can be relativized.

There are some other reducibilities. One of them is truth-table reducibility or abbreviated by tt-reducibility. If A is Turing reducible to B, then there exists an oracle Turing machine such that it computes A with the oracle B, but this machine may be undefined for an oracle other than B. If the reduction is total and defines a set for every oracle, then the reduction is called tt-reduction, and we say that A is *tt-reducible to* B. Some researchers say that A is *tt-computable relative to* B.

Uniform relativization is, roughly speaking, this tt-version of relativization. To distinguish them, we sometimes call Turing-reducibility version of relativization Turing relativization. Even if a notion is a tt-version of relativization of a known notion, the notion can be described using the terminology of tt-reduction in many cases. However, when relativizing a randomness notion, it is not appropriate to describe it via the reductions. We admitted that there are some types of relativization and named it the tt-version uniform relativization.

Uniform relativization of computable sets with an oracle B is nothing but the sets tt-reducible to B. In a more general setting, uniform relativization of a notion with an oracle B is defined by a total operator from oracles to the sets describing the notions and it is no longer a tt-reduction. We require uniformity for the operator, and that is the reason we call it uniform relativization.

Uniform relativization of Schnorr randomness behaves more naturally than Turing relativization of Schnorr randomness. This is where we found this relativization.

1.2 Algorithmic randomness

Randomness is a central notion in natural science. The theory of algorithmic randomness defines many randomness notions and studies their properties. For simplicity, from now on, the underlying space is the Cantor space 2^ω with the uniform measure μ on it.

Martin-Löf randomness (or ML-randomness) is the most studied notion and a subclass of 2^ω. An interesting result was shown for ML-randomness by van Lambalgen [16]: $X \oplus Y$ is ML-random if and only if X is ML-random and Y is ML-random Turing relative to X. Here, X, Y are infinite binary sequences and $X \oplus Y$ is the sequence alternating between X and Y. Intuitively, if a sequence is random, then the odd-numbered parts should be random, and the even-numbered parts should be random relative to the odd parts, and vice versa. This property should hold for every natural randomness notion and its suitable relativization.

For a randomness notion $R \subseteq 2^\omega$, consider a relativized version $R^A \subseteq 2^\omega$ with an oracle A. If van Lambalgen's theorem holds for this notion, then

$$X \oplus Y \in R \iff X \in R \text{ and } Y \in R^X.$$

Fix R, then the suitable relativization R^X is automatically determined for every $X \in R$. Hence, van Lambalgen's theorem can be used as a criterion of a natural relativization for a natural randomness notion.

Schnorr randomness is another natural randomness notion. This notion comes up naturally in computable measure theory. It turns out that van Lambalgen's theorem holds for Schnorr randomness with the uniform relativization, but not for Schnorr randomness with Turing relativization. This fact has many applications, and uniform relativization is a powerful tool in the study of Schnorr randomness.

Notice that Turing relativization is used in van Lambalgen's theorem for ML-randomness. For ML-randomness, its Turing relativization and uniform relativization are the same. Hence, uniform relativization of ML-randomness is not a new notion.

Another result relating to relativized randomness is *lowness*. A central result on this topic is the equivalence between lowness for ML-randomness and K-triviality. When giving a Schnorr-randomness version, we need uniform relativization of Schnorr randomness. This fact is another evidence that uniformly relativized Schnorr randomness is a fundamental notion.

In Section 2 we review some basic definitions and results. In Section 3 we introduce uniform relativization and give some related results. In Section 4 we gather some results relating to uniform lowness.

2 Preliminaries

2.1 Reduction

We follow standard notations in computability theory. For details, see e.g. [30, 4].

We identify a set $A \subseteq \mathbb{N}$ of natural numbers with a binary sequence $A \in 2^\omega = \{0,1\}^\omega$ by $n \in A \iff A(n) = 1$ for all n. Let $(\Phi_e)_{e\in\mathbb{N}}$ be a computable enumeration of all oracle Turing machines. The machine Φ can be seen as an operator with a partial domain from 2^ω to 2^ω as follows: For sets $A, B \in \mathbb{N}$, $B = \Phi^A$ is defined by $\Phi^A(n) = B(n)$ for every n. This Φ is called a *Turing reduction*. If this operator is total, then Φ is called a *tt-reduction*.

For sets $A, B \in \mathbb{N}$, A is *Turing reducible to* B, denoted by $A \leq_T B$, if there is a Turing reduction Φ such that $A = \Phi^B$. The set A is *tt-reducible to* B, denoted by $A \leq_{tt} B$, if there is a tt-reduction Φ such that $A = \Phi^B$.

For a computable set B, we have $A \leq_T B$ if and only if $A \leq_{tt} B$ if and only if A is computable. If $A \leq_{tt} B$, then clearly $A \leq_T B$. The converse does not hold. [1]

2.2 Randomness notions

We also follow standard notations in the theory of algorithmic randomness. For details, see e.g. [9, 25].

Cantor space 2^ω is the set of all infinite binary sequences equipped with the topology generated by the cylinder sets $[\sigma] = \{X \in 2^\omega \ : \ \sigma \prec X\}$ where $\sigma \in 2^{<\omega}$ is a finite binary sequence, and $\prec$ is the prefix relation. Let μ be the uniform measure on 2^ω defined by $\mu([\sigma]) = 2^{-|\sigma|}$ for every $\sigma \in 2^{<\omega}$.

A real $x \in \mathbb{R}$ is *computable* if there exists a computable sequence $(q_n)_n$ of rationals such that $|x - q_n| \leq 2^{-n}$ for all n. A real $x \in \mathbb{R}$ is *lower semicomputable* if there exists an increasing computable sequence $(q_n)_n$ of rationals such that

[1] Every noncomputable c.e. Turing degree contains a hypersimple set [26, Proposition III.3.13] while a hypersimple set is not tt-complete [26, Theorem III.3.10].

$x = \lim_{n\to\infty} q_n$. Every computable real is lower semicomputable, but there is a lower semicomputable real that is not computable.

An open set $U \subseteq 2^\omega$ is *c.e.* if there exists a computable sequence S of finite binary strings such that $U = \bigcup_{\sigma\in S}[\sigma]$. Notice that the measure of a c.e. open set is lower semicomputable, but not computable in general.

A *ML-test* is a uniform sequence $(U_n)_n$ of c.e. open sets such that $\mu(U_n) \leq 2^{-n}$ for all n. A set $X \in 2^\omega$ is *ML-random* if it passes each ML-test, that is, $X \notin \bigcap_n U_n$ for every ML-test $(U_n)_n$. A *Schnorr test* is a ML-test $(U_n)_n$ such that $\mu(U_n)$ is uniformly computable. A set X is *Schnorr random* if it passes each Schnorr test.

2.3 Computable analysis

To formalize uniform relativization of randomness notions, we use the terminology of computable analysis. For more details, see [32, 2].

Let X be a set. A *representation* of X is a surjective function $\delta :\subseteq \omega^\omega \to X$. For the real line, we usually consider the *Cauchy representation* $\rho_C :\subseteq \omega^\omega \to \mathbb{R}$ defined by

$$\rho_C(p_1, p_2, \cdots) = x \iff \lim_{n\to\infty} \nu_{\mathbb{Q}}(p_n) = x \text{ and } (\forall i < j)|\nu_{\mathbb{Q}}(p_i) - \nu_{\mathbb{Q}}(p_j)| \leq 2^{-i}$$

where $\nu_{\mathbb{Q}} :\subseteq \omega \to \mathbb{Q}$ is a computable notation of $\mathbb{Q}$. For the class $\mathcal{O}$ of all open sets on 2^ω, we usually consider the *inner representation* $\theta :\subseteq \omega^\omega \to \mathcal{O}$ defined by

$$\theta(p_1, p_2, \cdots) = \bigcup_n \nu(p_n)$$

where ν is a computable notation of the cylinder sets. For 2^ω, we use the identity $\mathrm{Id} :\subseteq \omega^\omega \to 2^\omega$ as a representation.

Let X be a set with a representation δ. If $\delta(p) = x \in X$ for $p \in \omega^\omega$, then p is called a *δ-name* of x. An element $x \in X$ is *δ-computable* if it has a computable δ-name. Then, $x \in \mathbb{R}$ is computable if and only if it is ρ_C-computable. An open set U is c.e. if and only if it is θ-computable.

For sets X_1, X_2 with representations δ_1, δ_2, a function $f :\subseteq X_1 \to X_2$ is *(δ_1, δ_2)-computable* if there is a computable function $g :\subseteq \omega^\omega \to \omega^\omega$ such that $f \circ \delta_1(p) = \delta_2 \circ g(p)$ for every $p \in \mathrm{dom}(\delta_1)$. Roughly speaking, given any δ_1-name p of $x \in X_1$, the function g computes a δ_2-name q of $f(x)$.

3 Uniform relativization

The goal of this section is to define uniform relativization of Schnorr randomness. As a warm-up, let us begin by defining uniform relativization of more basic objects.

3.1 Uniform relativization of c.e. sets

We do not try to define uniform relativization itself. Instead, we define uniform relativization of some notions.

A set $A \in 2^\omega$ is *computable uniformly relative to* B if there exists a tt-reduction Φ such that $A = \Phi^B$ or equivalently $A \leq_{tt} B$. The reduction can use B as an oracle, but should be total. This roughly means that the reduction cannot use a special property of B because the reduction should work for all oracles.

A set $A \subseteq \mathbb{N}$ is *c.e.* if there exists a Turing machine $\Phi :\subseteq \omega \to \omega$ such that $A = \mathrm{dom}(\Phi)$. The machine Φ can be seen as an operator. A set $A \subseteq \mathbb{N}$ is *c.e. relative to* B if there exists an oracle Turing machine $\Phi :\subseteq \omega \to \omega$ such that $A = \mathrm{dom}(\Phi^B)$. While the notion of tt-reduction requires the oracle Turing machine to be total, it is not the case in the relativization of c.e. sets: every oracle Turing machine sends each oracle Y to a set that is c.e. relative to Y, so every oracle Turing machine is defined everywhere in that sense. Thus, Turing relativization of c.e. sets is the same as uniform relativization of c.e. sets.

Recall that the use of tt-reduction has a computable bound. If $A \leq_{tt} B$ via Φ, then there exists a computable function $f : \omega \to \omega$ such that the oracle use of $\Phi^X(n)$ is bounded by $f(n)$. In the computation of $\mathrm{dom}(\Phi^B)$ we do not have such a bound. Uniform relativization is similar to tt-reduction, but it is not appropriate to identify them.

3.2 Uniform relativization of c.e. open sets

Let us turn to c.e. open sets. An open set $U \subseteq 2^\omega$ is *c.e.* if it is θ-computable. An open set U is *c.e. relative to* $B \in 2^\omega$ if there is a (Id, θ)-computable function $f :\subseteq 2^\omega \to \mathcal{O}$ such that $f(B) = U$. The function f can be partial but we require $B \in \mathrm{dom}(f)$. This is the usual Turing relativization. Notice that the function $f(X)$ produces a c.e. set for every input $X \in 2^\omega$, so again uniform relativization of c.e. openness is the same as its Turing relativization.

We consider the notion of c.e. openness with the measure $\leq 2^{-n}$ for $n \in \mathbb{N}$. This strange notion comes up in the relativization of Martin-Löf randomness. An open set $U \subseteq 2^\omega$ is *c.e.* and has measure $\leq 2^{-n}$ relative to $B \in 2^\omega$ if there is a (Id, θ)-computable function $f :\subseteq 2^\omega \to \mathcal{O}$ such that $f(B) = U$ and $\mu(U) \leq 2^{-n}$. In this case, the measure of $\mu(f(X))$ may be larger than 2^{-n} for some oracle $X \in 2^\omega$. However, we can modify f by restricting the enumeration of the cylinder sets as long as its measure is $\leq 2^{-n}$. This modified function $\hat{f}$ is also computable, the measure of $\hat{f}(X)$ is $\leq 2^{-n}$ for each $X \in 2^\omega$, and $\hat{f}(B) = U$. Hence, again its uniform relativization is the same as its Turing relativization.

Finally, we consider the notion of c.e. openness with a computable measure. This notion corresponds to the relativization of Schnorr randomness. An open set $U \subseteq 2^\omega$ is *c.e.* and has a computable measure Turing relative to $B \in 2^\omega$ if there are a (Id, θ)-computable function $f :\subseteq 2^\omega \to \mathcal{O}$ and a (Id, ρ_C)-computable function $g :\subseteq 2^\omega \to \mathbb{R}$ such that $f(B) = U$ and $g(B) = \mu(U)$. Notice that f, g can be partial but we should have $B \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$.

In this case, we can not extend f, g to be total by any computable modification. Let us give a counterexample. Let $A, B \subseteq \mathbb{N}$ be sets such that $A \leq_T B$ but $A \not\leq_{tt} B$. Define $U = \bigcup_{k \in A} [0^k 1]$. Since $A \leq_T B$, the open set U is c.e. Turing relative to B and its measure is computable Turing relative to B.

Suppose that there exist a total (Id, θ)-computable function $f :\subseteq 2^\omega \to \mathcal{O}$ and a total (Id, ρ_C)-computable function $g :\subseteq 2^\omega \to \mathbb{R}$ such that $\mu(f(Y)) = g(Y)$ for every $Y \in 2^\omega$ and $f(B) = U$. Consider the following reduction $\Phi :\subseteq 2^\omega \to 2^\omega$ with an input $Y \in 2^\omega$. For each $n \in \mathbb{N}$, enumerate the inner cylinders of $f(Y)$ until the measure larger than $g(Y) - 2^{-n-1}$. If the intersection between the finite approximation and $[0^n 1]$ is not empty, then $\Phi^Y(n)$ outputs 1. Otherwise, $\Phi^Y(n)$ outputs 0.

Since $\mu(f(Y)) = g(Y)$ for every $Y \in 2^\omega$, the reduction can find the finite approximation for every n and Φ is total. Suppose $Y = B$ and $n \in A$. Then, the finite approximation U_n of $f(B)$ should intersect with $[0^n 1]$, otherwise $\mu(U_n) > \mu(g(Y)) - 2^{-n-1} = \mu(U) - 2^{-n-1}$ and $\mu(U_n \cup [0^n 1]) = \mu(U_n) + 2^{-n-1} > \mu(U)$, which contradicts with $U_n \cup [0^n 1] \subseteq U$. Hence, $\Phi^B(n) = 1 = A(n)$. Suppose $Y = B$ and $n \notin A$. Then, the finite approximation U_n of $f(B)$ can not intersect with $[0^n 1]$ because U_n is the inner approximation of $U = \bigcup_{k \in A} [0^k 1]$. Hence, $\Phi^B(n) = 0 = A(n)$. This contradicts with $A \not\leq_{tt} B$.

Now we know that uniform relativization of c.e. openness with a computable measure is different from its Turing relativization.

3.3 Uniform relativization of Schnorr randomness

We are now ready to define uniform relativization of Schnorr randomness, or abbreviated by uniform Schnorr randomness. The definition is complicated, but the idea is the same as the basic notions defined in the above.

Definition 1 ([23]). *A* uniform Schnorr test *is a pair of computable functions f, g satisfying the follows:*

1. *$f : 2^\omega \times \omega \to \mathcal{O}$ is $(\mathrm{Id}, \mathrm{Id}_\omega, \theta)$-computable with $\mu(f(X, n)) \leq 2^{-n}$ for all $X \in 2^\omega$ and $n \in \omega$.*
2. *$g : 2^\omega \times \omega \to \mathbb{R}$ is $(\mathrm{Id}, \mathrm{Id}_\omega, \rho_C)$-computable such that $g(X, n) = \mu(f(X, n))$ for all $X \in 2^\omega$ and $n \in \omega$.*

Here, $\mathrm{Id}_\omega : \omega \to \omega$ is the identity function on ω. A set $A \in 2^\omega$ is Schnorr random uniformly relative to B *if $A \notin \bigcap_n f(B, n)$ for each uniform Schnorr test $\langle f, g \rangle$.*

Each Schnorr test uniformly relative to B is a Schnorr test Turing relative to B. Thus, each Schnorr random Turing relative to B is Schnorr random uniformly relative to B. Hence, uniform relativized randomness is weaker than Turing relativized randomness.

We need this relativization for van Lambalgen's theorem for Schnorr randomness to hold. For sets $A, B \subseteq \mathbb{N}$, let $A \oplus B = \{2n \ : \ n \in A\} \cup \{2n + 1 \ : \ n \in B\}$.

Theorem 1 ([23]). *The set $A \oplus B$ is Schnorr random if and only if A is Schnorr random and B is Schnorr random uniformly relative to A.*

Theorem 2 ([18, 33] and [25, Remark 3.5.22]). *Van Lambalgen's theorem fails for Schnorr randomness with Turing relativization.*

In particular, uniform relativization of Schnorr randomness is different from its Turing relativization.

Notice that, if A is Schnorr random and B is Schnorr random Turing relative to A, then $A \oplus B$ is Schnorr random. This is because uniformly relativized Schnorr randomness is weaker than Turing relativized Schnorr randomness. For the other direction, assume that A is Schnorr random and B is covered by a Schnorr test relative to A. One needs uniformity or totality of the test to construct a Schnorr test covering $A \oplus B$. Uniform relativization naturally comes up when looking at the proofs of van Lambalgen's theorem.

3.4 Other characterizations

In the above, we defined uniform Schnorr randomness via tests. Schnorr randomness has characterizations by martingales, computable measure machines, and integral tests. We can also characterize uniform Schnorr randomness by them. The proofs are straightforward, but we need to check that everything works uniformly in oracles. We give the definitions to look at how to uniformly relativize these notions.

A *martingale* is a function $d : 2^{<\omega} \to \mathbb{R}^+$ such that $2d(\sigma) = d(\sigma 0) + d(\sigma 1)$ for every $\sigma \in 2^{<\omega}$ where $\mathbb{R}^+$ is the set of all nonnegative reals. A set $X \in 2^\omega$ is ML-random if and only if $\sup_n d(X \upharpoonright n) < \infty$ for all left-c.e. martingales d. A set $X \in 2^\omega$ is Schnorr random if and only if $d(X \upharpoonright n) < f(n)$ for at most finitely many n for every computable martingale d and every computable order f. These are classical results by Schnorr [29, 28]. Here, an order is an unbounded nondecreasing function from ω to ω. Franklin and Stephan [10] observed that X is not Schnorr random if and only if there is a computable martingale d and a computable function f such that $(\exists^\infty n) d(X \upharpoonright f(n)) \geq n$.

A *uniformly computable martingale* is a computable map $d : 2^\omega \times 2^{<\omega} \to \mathbb{R}^+$ such that $d^Z := d(Z, \cdot)$ is a martingale for every $Z \in 2^\omega$. A set X is Schnorr random uniformly relative to A if and only if $d^A(X \upharpoonright n) < f(n)$ for almost all n for each uniformly computable martingale d and a computable order f if and only if $d^A(X \upharpoonright h(n)) < n$ for almost all n for each uniformly computable martingale d and a strictly increasing computable function h [19]. We can replace the computable order f above with $\hat{f}^A$ such that $\hat{f} : 2^\omega \times \omega \to \omega$ is a computable function and $\hat{f}^Z$ is an order for each $Z \in 2^\omega$. This is because, for such $\hat{f}$, we can find a computable order f such that $\hat{f}^Z(n) \geq f(n)$ for each $n \in \mathbb{N}$ and each $Z \in 2^\omega$ by compactness of 2^ω.

Franklin and Stephan [10] defined tt-Schnorr random set X relative to A as a set such that there are no martingale $d \leq_{tt} A$ and no function $g \leq_{tt} A$ such that $(\exists n) d(X \upharpoonright h(n)) \geq n$. We can replace $h \leq_{tt} A$ with a computable function h [10, Remark 2.4]. This notion is equivalent to Schnorr randomness uniformly relative to A [23, Proposition 6.1]. However, there are some subtle points to note in the tt-relativization. See Section 6 in [23] for details.

Let $M :\subseteq 2^{<\omega} \to 2^{<\omega}$ be a Turing machine. The *measure* (or halting probability) of M is $\sum_\sigma \{2^{-|\sigma|} \ : \ M(\sigma)\downarrow\}$. The measure of a prefix-free Turing machine is less than or equal to 1 by Kraft's inequality. The measure of a universal prefix-free Turing machine U is called Chaitin's omega, denoted by Ω_U, which is ML-random [3], hence not computable. A prefix-free Turing machine with a computable measure is called a *computable measure machine*. A set X is Schnorr random if and only if $K_M(X \upharpoonright n) > n - O(1)$ for every computable measure machine M [7].

An oracle prefix-free Turing machine $M :\subseteq 2^\omega \times 2^{<\omega} \to 2^{<\omega}$ is a *uniformly computable measure machine* if the maps $X \mapsto \sum_\sigma \{2^{-|\sigma|} \ : \ M(X,\sigma)\downarrow\}$ is a total computable function. A set X is Schnorr random uniformly relative to A if and only if $K_{M^A}(X \upharpoonright n) > n - O(1)$ for every uniformly computable measure machine M (essentially due to [19]).

An *integral test* is an integrable nonnegative lower semicomputable function $f : 2^\omega \to \mathbb{R}^+$. A set $X \in 2^\omega$ is ML-random if and only if $f(X) < \infty$ for each integral test, which is by Levin: see e.g. [17, Subsection 4.5.6, 4.7]. A set $X \in 2^\omega$ is Schnorr random if and only if $f(X) < \infty$ for each nonnegative lower semicomputable function $f : 2^\omega \to \mathbb{R}^+$ such that $\int f \ d\mu$ is a computable real [20]. Such a function is called a *Schnorr integral test*.

A *uniform Schnorr integral test* is a lower semicomputable function $f : 2^\omega \times 2^\omega \to \mathbb{R}^+$ such that $X \mapsto \int f(X,Z)\mu(dZ)$ is a computable function from 2^ω to $\mathbb{R}$. The first component is for oracles and the second for the tested sets A set Y is Schnorr random uniformly relative to X if and only if $f(X,Y) < \infty$ for each uniform Schnorr integral test f [23, Proposition 4.1].

3.5 Related work

Van Lambalgen's theorem for uniform Schnorr randomness was further generalized to noncomputable measures [27], and was used in the study of Schnorr reducibility and total-machine reducibility [22]. Van Lambalgen's theorem for uniform relativization of computable randomness holds in a weaker form [23, Theorem 5.1]. Van Lambalgen's theorem for uniform Kurtz randomness was studied in [13]. Van Lambalgen's theorem for Demuth randomness was studied in [5], where they used ”partial relativization.”

4 Uniform lowness

Another topic relating to relativized randomness is lowness. First, we recall some results on lowness for ML-randomness. Then, we see that uniform lowness was needed to give a Schnorr-randomness version.

4.1 Characterization of triviality via lowness

Many randomness notions have characterizations via complexity. The Levin-Schnorr theorem says that $A \in 2^\omega$ is ML-random if and only if $K(A \upharpoonright n) >$

$n - O(1)$ where K is the prefix-free Kolmogorov complexity. Roughly speaking, a set is random if the complexities of its initial segments are high. Thus, the complexity is a measure of randomness. A set $A \in 2^\omega$ is K-reducible to $B \in 2^\omega$ if $K(A \upharpoonright n) < K(B \upharpoonright n) + O(1)$. This is one formalization of saying that A is not more random than B. The class of K-trivial sets is the bottom degree of this reducibility. A set $A \in 2^\omega$ is *K-trivial* if $K(A \upharpoonright n) \leq K(n) + O(1)$. Obviously, every computable set is K-trivial. In contrast, there is a noncomputable K-trivial set.

Interestingly, K-triviality can be characterized by lowness. A set $A \in 2^\omega$ is *low for ML-randomness* if every ML-random set Turing relative to A is already (unrelativized) ML-random. This means that the set A can not derandomize any ML-random set. These notions coincide, that is, a set $A \in 2^\omega$ is K-trivial if and only if A is low for ML-randomness. For details on this topic, see e.g. [24].

We have Schnorr-randomness counterparts of these notions as follows.

Definition 2 ([7]). *A set $A \in 2^\omega$ is* Schnorr reducible to $B \in 2^\omega$ *denoted by $A \leq_{Sch} B$ if, for every computable measure machine M, there exists a computable measure machine N such that $K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + O(1)$. A set $A \in 2^\omega$ is called* Schnorr trivial *if $A \leq_{Sch} \emptyset$.*

This notion can be characterized by uniform lowness for Schnorr randomness. A set $A \in 2^\omega$ is called *uniformly low for Schnorr randomness* if every Schnorr random set uniformly relative to A is already Schnorr random.

Theorem 3 (essentially due to [10]). *A set $A \in 2^\omega$ is Schnorr trivial if and only if A is uniformly low for Schnorr randomness.*

Since there is a Turing-complete Schnorr trivial set [8], some Schnorr trivial sets are not Turing low for Schnorr randomness. Hence, uniform lowness and Turing lowness for Schnorr randomness are different.

4.2 Other characterizations

The class of K-trivial sets has many characterizations, and so does the class of Schnorr trivial sets.

The first one is by traceability. A *trace* is a sequence $\{T_n\}$ of sets. A trace for a function f is a trace $\{T_n\}$ with $f(n) \in T_n$ for all n. For a function h, a trace $\{T_n\}$ is h-bounded if $|T_n| \leq h(n)$ for all n. A set A is *computable tt-traceable* if there is a computable order h such that all functions $f \leq_{tt} A$ are traced by an h-bounded computable trace. Roughly speaking, the values computable from traceable sets have limited possibilities. Many variants were studied in Hölzl and Merkle [12].

Franklin and Stephan [10] showed that uniform lowness of Schnorr randomness is equivalent to computable tt-traceability. There is no counterpart for ML-randomness. This result is a modification of the one that Turing lowness of Schnorr randomness is equivalent to computable (Turing) traceability [31, 15].

The next one is by lowness for machines. A set A is called *low for* K if $K(n) \leq K^A(n) + O(1)$. This means that the set A can not compress n more than without it. In fact, a set is K-trivial if and only if it is low for K.

A Schnorr-randomness version is as follows. We say that a set A is *uniformly low for* computable measure machines if, for every uniformly computable measure machine M, there exists a computable measure machine N such that $K_N(n) \leq K_{M^A}(n) + O(1)$. Then, uniform lowness for computable measure machines is equivalent to computable tt-traceability [19], hence to Schnorr triviality. The proof was given by straightforward modification of the fact that Turing lowness for computable measure machines is equivalent to computable traceability [6].

The class of K-trivial sets also has a base-type characterization. A set A is a *base for* ML-randomness if there exists a ML-random set X relative to A such that $A \leq_T X$. Notice that each computable set is a base for ML-randomness. If A has much information, the class of ML-random sets relative to A is so small that we can not find such a set in the Turing degrees above A.

Its Schnorr-randomness version is not straightforward. See the discussion in [10, Section 6]. We say that a set A is a *base for Schnorr randomness* if there is no $X \geq_T A$ such that X is Schnorr random Turing relative to A. Franklin, Stephan, and Yu [11] showed that this is equivalent to saying that the set A does not compute the halting problem.

One adaptation is as follows. A set A is a *tt-base for uniformly computable martingales* if, for each uniformly computable martingale d, there exists a set $X \geq_{tt} A$ such that $\sup_n d^A(X \upharpoonright n) < \infty$. The last condition roughly means that X is computably random uniformly relative to A only for this d. It turns out that Schnorr triviality is equivalent to being a tt-base for uniformly computable martingales [21, Theorem 6.4].

4.3 Related work

Decidable prefix-free machines also characterize ML-randomness and Schnorr randomness [1]. Schnorr reducibility can be characterized by complexity for prefix-free decidable machines by adding a computable order. We write $A \leq_{wdm} B$ if, for each decidable prefix-free machine M and a computable order g, there exists a decidable prefix-free machine N such that $K_N(A \upharpoonright n) \leq K_M(B \upharpoonright n) + g(n) + O(1)$. In fact, $A \leq_{wdm} B$ if and only if $A \leq_{Sch} B$ [21, Theorem 3.5]. In particular, Schnorr triviality has a characterization by decidable prefix-free machines.

Schnorr triviality is also equivalent to not totally i.o. complex [12], which is a characterization by total machines.

The equivalence between lowness for ML-randomness and lowness for K was strengthened to the equivalence between $\leq_{LR}$ and $\leq_{LK}$ [14]. Its uniform Schnorr-randomness version was proved in [21, Theorem 5.1] and Turing relativized Schnorr-randomness version in [22].

Computable traceability was characterized by order-lowness for prefix-free decidable machines [1, Theorem 24]. Recall that computable traceability is equivalent to Turing lowness for Schnorr randomness.

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