Subclasses of the weakly computable reals

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Real closed field

We consider a subclass $\mathcal{F} \subseteq \mathbb{R}$.

$\mathcal{F}$ is called a real closed field if it is closed under

- addition, subtraction, multiplication, division,
- taking real roots of polynomials with coefficients in $\mathcal{F}$

Examples

Example 1

1. $\mathbb{R}$
2. the real algebraic numbers (real roots of polynomials of rational coefficients)
3. the computable reals (see Pour-El and Richards 1989)
4. the weakly computable reals (Ng 2005, Raichev 2005)
5. the non-random weakly computable reals (Miller 2017)
6. the primitive recursive reals (due to Peter Hertling, see Selivanov and Selivanova 2021)

Goal

Definition 2

Let α, β be weakly computable reals. Then, α is strongly Solovay reducible to β, denoted by $\alpha \ll_S \beta$, if, for every $c > 0$, there exist comp. $(a_n) \to \alpha$ and $(b_n) \to \beta$ such that

$$|a_n - \alpha| \leq c(|b_n - \beta| + 2^{-n})$$

for all n.

Theorem 3

For any $\beta \in \mathbb{R}_{wc}$, $\{\alpha \ : \ \alpha \ll_S \beta\}$ is a real closed field.

Here, $\alpha \ll_S \Omega$ if and only if α is a non-random weakly computable real.

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Definition

First, we review some definitions and related results.

Weakly computable reals

A real $\alpha \in \mathbb{R}$ is computable if there is a computable sequence $(a_n)_n$ of rationals such that $|a_n - a_{n-1}| \leq 2^{-n}$ for all $n \geq 1$ and $\lim_n a_n = \alpha$.

A real $\alpha \in \mathbb{R}$ is weakly computable if there is a computable sequence $(a_n)_n$ of rationals such that $\sum_n |a_n - a_{n-1}| < \infty$ and $\lim_n a_n = \alpha$.

A real $\alpha \in \mathbb{R}$ is left-c.e. if there is a computable sequence $(a_n)_n$ of rationals such that $a_{n-1} \leq a_n$ for all $n \geq 1$ and $\lim_n a_n = \alpha$.

A real is weakly computable if and only if it is the difference of two left-c.e. reals (Ambos-Spies, Weihrauch, and Zheng 2000).

Thus weakly computable reals are sometimes called d.c.e. reals.

Extended Solovay reducibility

Definition 4 (Rettinger and Zheng 2005, after Solovay 1975)

$\alpha \in \mathbb{R}_{wc}$ is (extended) Solovay reducible to $\beta \in \mathbb{R}_{wc}$ (denoted by $\alpha \leq_S \beta$) if there are computable sequence $(a_n)_n$ and $(b_n)_n$ of rationals such that

$$\lim_n a_n = \alpha,\ \lim_n b_n = \beta,\ (\exists c)(\forall n)(|a_n - \alpha| \leq c(|b_n - \beta| + 2^{-n})$$

Intuitively, we can compute, from a good approximation of β, a good approximation of α. However, the approximation b_n may be too good. Thus, we add the error 2^{-n}.

Randomness

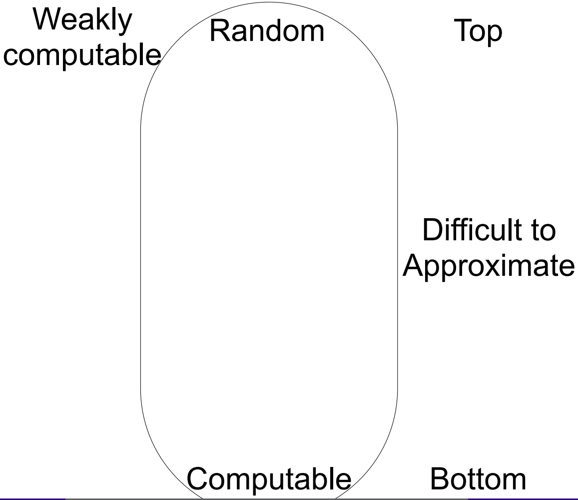
difficult to approximate $\approx$ random

Theorem 5 (Rettinger and Zheng 2005, after Chaitin 1976, Solovay 1975, Calude et al. 1998, Kučera and Slaman 2001)

Any Martin-Löf random real in $\mathbb{R}_{wc}$ is Solovay complete.

- Any random left-c.e. real is Solovay complete in left-c.e. reals.
- Any random weakly computable real is left-c.e. or right-c.e.
- Any weakly computable real is Solovay reducibile to Ω.

Summary


Weakly
computable
Random
Top
Difficult to
Approximate
Computable
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Extension to reducibility

Theorem 6

Let $\beta \in \mathbb{R}_{wc}$. *Then,*

$$S(\leq \beta) = \{\alpha \in \mathbb{R}_{wc} \ : \ \alpha \leq_S \beta\}$$

is a real closed field.

Locally Lipschitz

Definition 7

$f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz if, for $x \in \mathrm{dom}(f)$, there is a neighborhood U of x and a Lipschitz-constant L such that

$$(\forall \mathbf{u}, \mathbf{v} \in U)|f(\mathbf{u}) - f(\mathbf{v})| \leq L \cdot |\mathbf{u} - \mathbf{v}|)$$

where $|\mathbf{u} - \mathbf{v}| = \sum_{i=1}^{n} |u_i - v_i|$.

$S(\leq \beta)$ is closed under a locally Lipschitz computable function: If $\alpha \leq_S \beta$, then $f(\alpha) \leq_S \beta$.

Proof ingredients

Rettinger and Zheng (2005) have already shown that $S(\leq \beta)$ is a field.

McNicholl (2008) showed that the implicit function for continuously differentiable computable functions is computable.

Following Raichev (2005), the combination with these results shows the result above.

Another extension

Miller's result states that the non-random weakly computable reals form a real closed field. Another formulation is that

$$\{\alpha \in \mathbb{R}_{wc} \ : \ \alpha <_S \Omega\}$$

forms a real closed field.

One interpretation is that one can not construct a random real from non-random reals by taking real roots of polynomials.

Another extension impossible

For any non-random $\beta \in \mathbb{R}_{wc}$, $\{\alpha \in \mathbb{R}_{wc} \ : \ \alpha <_S \beta\}$ is not a real closed field.

Theorem 8 (Demuth 1975)

If α, β are left-c.e. reals such that $\alpha + \beta$ is random, then at least one of α, β is random.

Theorem 9 (Downey, Hirschfeldt, and Nies 2002)

If α is non-random left-c.e. real. Then, there are left-c.e. reals β, γ such that $\beta, \gamma <_S \alpha$ and $\beta + \gamma = \alpha$.

Strong Solovay reducibility

Let α, β be positive left-c.e. reals.

$\alpha \ll_S \beta$ if one of the following equivalent conditions holds:

1. For any $c > 0$, there exist $(a_n) \uparrow \alpha$ and $(b_n) \uparrow \beta$ such that $\alpha - a_n < c(\beta - b_n)$ for all n.
2. $c\beta - \alpha$ is a left-c.e. real for any $c > 0$.
3. $\dfrac{c\beta}{\alpha}$ is a left-c.e. real for any $c > 0$.

Compare the equation in Miller (2017):

$$\partial\alpha = \frac{\partial\alpha}{\partial\Omega} = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s} = \inf\{c \in \mathbb{Q} \ : \ c\Omega - \alpha \text{ is left-c.e.}\}.$$

Strong Solovay reducibility

Let α, β be left-c.e. reals. Then,

$$\alpha \ll_K \beta \Rightarrow \alpha \ll_S \beta \Rightarrow \alpha <_S \beta \Rightarrow \alpha \leq_K \beta.$$

where $\alpha \ll_K \beta$ if $\lim_n (K(\beta \upharpoonright n) - K(\alpha \upharpoonright n)) = \infty$.
Furthermore, the converses do not hold. For the first one, there exists a non-random left-c.e. real α such that $\liminf_n (K(\Omega \upharpoonright n) - K(\alpha \upharpoonright n)) < \infty$. For the second one, consider a non-random β and $\alpha_1, \alpha_2 <_S \beta$ such that $\alpha_1 + \alpha_2 = \beta$.

Strong Solovay reducibility

Then, if $\alpha \ll_S \gamma$ and $\beta \ll_S \gamma$, then $\alpha + \beta \ll_S \gamma$. Furthermore,

$$\alpha \ll_S \Omega \iff \alpha <_S \Omega,$$

$$\alpha \ll_S \beta <_S \Omega \Rightarrow \alpha <_S \beta$$

Strong Solovay for weakly computable reals

Definition 10

Let α, β be weakly computable reals. Then, α is strongly Solovay reducible to β, denoted by $\alpha \ll_S \beta$, if, for every $c > 0$, there exist comp. $(a_n) \to \alpha$ and $(b_n) \to \beta$ such that

$$|a_n - \alpha| \leq c(|b_n - \beta| + 2^{-n})$$

for all n.

Theorem 11

For any $\beta \in \mathbb{R}_{wc}$, $\{\alpha \ : \ \alpha \ll_S \beta\}$ is a real closed field.

Here, $\alpha \ll_S \Omega$ if and only if α is a non-random weakly computable real.

Some properties

- If $\alpha_1 \equiv_S \alpha_2$, $\beta_1 \equiv_S \beta_2$, and $\alpha_1 \ll_S \beta_1$, then $\alpha_2 \ll_S \beta_2$.
- For left-c.e. reals, the two definitions coincide.
- $\alpha \ll_S \alpha$ if and only if α is computable.

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rK-reducibility

Definition 12 (Downey, Hirschfeldt, and LaForte 2004)

$\alpha \leq_{rK} \beta$ if

$$K(\alpha \upharpoonright n \mid \beta \upharpoonright n) = O(1).$$

Theorem 13 (Raichev 2005)

For each $\beta \in 2^\omega$,

$$\{\alpha \ : \ \alpha \leq_{rK} \beta\}$$

forms a real closed field.

K-trivial

Theorem 14

The class of K-trivial reals is a subclass of weakly computable reals and forms a real closed field.

Theorem 15 (Figueira, Stephan, and Wu 2006)

There is a universal machine U and an integer m such that for every K-trivial real $R \in [0, 2^{-m}]$ there is a co-c.e. set X with $R = \Omega_U[X] = \sum_{p:U(p)\downarrow\in X} 2^{-|p|}$.

Theorem 16 (Downey, Hirschfeldt, Nies, and Stephan 2003)

If α, β are K-trivial, then so is $\alpha + \beta$.

Extension to weakly computable reals

Let α, β be left-c.e. reals. Then,

$$\begin{aligned}
\alpha \ll_K \beta &\Rightarrow (\forall (a_n) \uparrow \alpha)(\forall (b_n) \uparrow \beta) \lim_n \frac{\alpha - a_n}{\beta - b_n} = 0 \\
&\Rightarrow (\exists (a_n) \uparrow \alpha)(\exists (b_n) \uparrow \beta) \lim_n \frac{\alpha - a_n}{\beta - b_n} = 0 \\
&\Rightarrow \alpha \ll_S \beta \Rightarrow \alpha <_S \beta
\end{aligned}$$

There are some other possibilities for the combination and the order of $\forall$ and $\exists$.

Can we extend to the weakly computable reals?

Straightforward extension is impossible

Fact 17

There are $\alpha \in \mathbb{R}_{wc}$ *and* $\beta \in \mathbb{R}_{lce}$ *such that*

$$\alpha \ll_K \beta, \quad \alpha \not\leq_S \beta.$$

Let $\Omega_{1/2} = \sum_{p:U(p)\downarrow} 2^{-2|p|}$. Then, the effective Hausdorff dimension is $1/2$, and the number of the changes can be bounded, from which we can construct a desired weakly computable real α.

Strong K to Strong Solovay

$\ll_K$ implies $\ll_S$ for left-c.e. reals

The key observation is that all following values have roughly the same complexity:

- $\alpha \upharpoonright n$
- $\min\{s \ : \ \alpha - a_s < 2^{-n}\}$
- $\max\{s \ : \ \alpha - a_s > 2^{-n}\}$

If β converges faster than α, then one can compute an initial segment of β from that of α.

This argument does not work for weakly computable reals.

Monotonically computable reals

Definition 18 (Rettinger et al. 2001)

$\alpha \in \mathbb{R}$ is k-monotonically computable if, $(a_n) \to \alpha$ such that

$$k \cdot |\alpha - a_n| \geq |\alpha - a_m|$$

for any $m > n$.

α is m.c. if it is k-m.c. for some $k > 0$.

Once one of approximations is close, then the remaining approximations can not be too far.

Monotonically computable reals

$\alpha \in \mathbb{R}$ is 1-m.c. if and only if it is left-c.e. or right-c.e. (Rettinger et al. 2001)

Any m.c. real is weakly computable. (Rettinger et al. 2001)

For $c_1 > c_2 > 1$, there exists a real such that it is c_1-m.c. but not c_2-m.c. (Rettinger and Zheng 2003)

Strong K implies Strong Solovay

Fact 19

Let α, β be m.c. reals. Then,

$$\alpha \ll_K \beta \Rightarrow \alpha \ll_S \beta.$$

Thank you for listening.

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