Solovay reducibility and signed-digit representation

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15 June, 2023
Form omega to Omega @ NUS

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Goal

We give some new characterizations of Solovay reducibility for weakly computable reals.

1. By Turing reduction with bounded use with respect to the signed-digit representation.
2. By upper and lower semi-computable Lipschitz functions.

"Solovay reducibility" is well-behaved even outside of left-c.e. reals!!

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Computability of reals

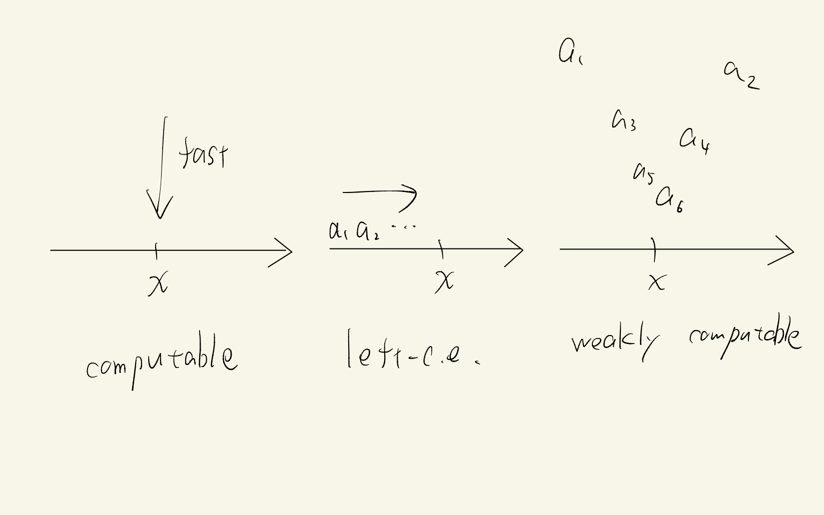
$\alpha \in \mathbb{R}$ is computable if $\exists (a_n)_n$, comp, $|a_n - \alpha| < 2^{-n}$ for all $n \in \omega$.

$\alpha \in \mathbb{R}$ is left-c.e. if $\exists (a_n)_n$, comp., increasing, converging to α

$\alpha \in \mathbb{R}$ is weakly computable (d.c.e., d.l.c.e.)
if $\exists (a_n)_n$, comp. $\sum_n |a_{n+1} - a_n| < \infty$, converging to α,
or equivalently if $\alpha = \beta - \gamma$ for left-c.e. reals β, γ.

$$\mathbf{EC} \subsetneq \mathbf{LC} \subsetneq \mathbf{WC}.$$

Picture of computability



Solovay reducibility for left-c.e. reals

Definition 1 (Solovay 1970s)

Let $\alpha, \beta \in \mathbf{LC}$. α is Solovay reducible to β, denoted by $\alpha \leq_S \beta$, if $\exists (a_n)_n, (b_n)_n$, comp., increasing, converging to α, β and $\exists c \in \omega$ such that

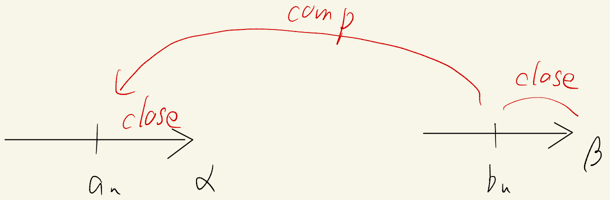
$$\alpha - a_n < c(\beta - b_n).$$

If one has a good approximation of β from below, then one can compute a good approximation of α.

Theorem 2 (Kučera and Slaman 2001 with some other results)

A left-c.e. real β is Martin-Löf random if and only if it is Solovay complete in left-c.e. reals, that is, $\alpha \leq_S \beta$ for all left-c.e. reals α.

Picture of Solovay reducibility



Solovay reducibility for weakly computable reals

Definition 3 (Zheng and Rettinger 2004)

Let $\alpha, \beta \in \mathbf{WC}$. α is Solovay reducible to β, denoted by $\alpha \leq_S \beta$, if $\exists (a_n)_n, (b_n)_n$, comp., converging to α, β and $\exists c \in \omega$ such that

$$|\alpha - a_n| < c(|\beta - b_n| + 2^{-n}).$$

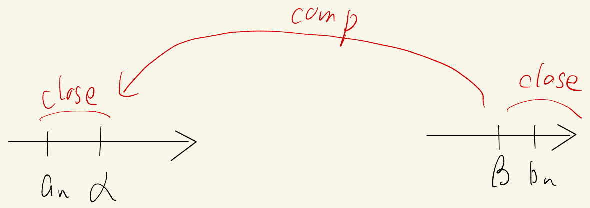
$(a_n)_n, (b_n)_n$ need not be increasing
The definition also works for limit computable reals (=computably approximable reals), but we focus on weakly computable reals for simplicity.

Proposition 4 (Rettinger and Zheng 2005)

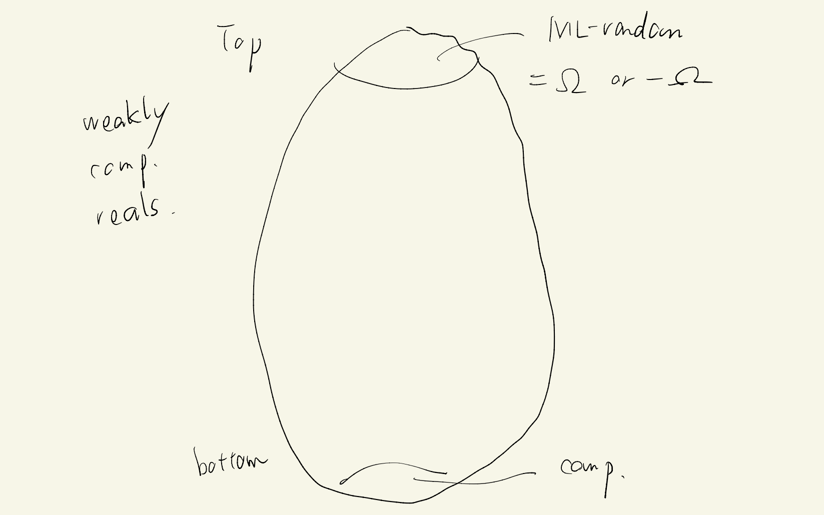
If a weakly computable real is ML-random, then it is left-c.e. or right-c.e.
A weakly computable real β is Martin-Löf random if and only if it is Solovay complete in weakly computable reals.

Thus, Solovay is complete if and only if left-c.e. ML-random (Ω) or right-c.e. ML-random($-\Omega$).

Picture of Solovay reducibility



Picture of Solovay reducibility



Question

The original Solovay reducibility is well-behaved within left-c.e. reals.
The Solovay reducibility by Zheng and Rettinger is well-behaved within weakly computable reals.

Question 5

Is there a reducibility such that

1. it has many good properties like Solovay reducibility,
2. it is well-behaved for all reals.

cL-reducibility

It would be desirable that Solovay reducibility can be characterized via Turing use bounds like tt and wtt.

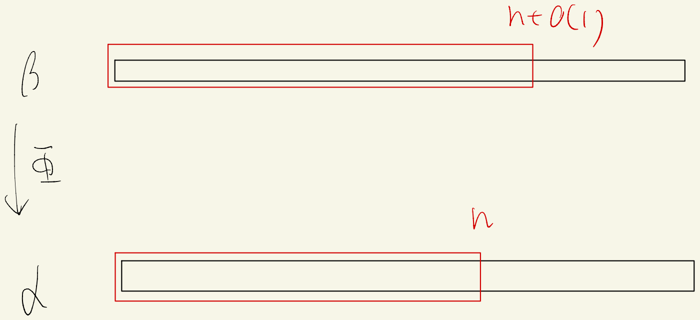
Definition 6 (Downey, Hirschfeldt, and LaForte 2004)

Let $\alpha, \beta \in 2^\omega$. Then, α is computably Lipschitz reducible to β, denoted by $\alpha \leq_{cL} \beta$, if $\exists\Phi$: Turing functional s.t.

- $\alpha = \Phi(\beta)$,
- $\mathrm{use}(\Phi, \beta, n) \leq n + O(1)$.

Solovay reducibility requires us to compute 2^{-n}-approximation of α from $2^{-n-O(1)}$-approximation of β. In this sense, these reducibilities are similar but, unfortunately, incomparable (see Theorem 9.1.6 and 9.10.1 in Downey and Hirschfeldt 2010).

Picture of Solovay reducibility



signed-digit representation

The main reason for the difference between Solovay reducibility and cL-reducibility is that the reals change continuously, while the binary sequences change discretely. A similar problem occurs in computable analysis, where we use the signed-digit representation for reals.

Theorem 7 (Kumabe, M., Suzuki)

Let $\alpha, \beta \in \mathbf{WC}$. $\alpha \le_S \beta$ if and only if it $\exists g$: partial comp. func. s.t.

- $\alpha = g(\beta)$,
- *g is (ρ, ρ)-computable with use bound $H(n) = n + O(1)$,*

where ρ is the signed-digit representation.

We will define the sd-representation later.
Replacing the binary representation in cL-reducibility with the sd-representation characterizes Solovay reducibility!

Solovay reducibility for all reals

This feature is pleasing in several ways.

- We have Solovay reducibility for all reals by redefining it via the use bound w.r.t. sd-representation.
- The condition uses use bound like many other reducibilities in computability theory.
- This clarifies the relation between Solovay reducibility and Lipschitz functions.

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Definition of sd-representation

The usual binary representation:

$$p \in 2^\omega,\ \rho_{bin}(p) = \sum_{n=0}^{\infty} p(n)2^{-n-1} \in [0,1].$$

Even if $\alpha \in [a,b]$ with $b - a < 2^{-n}$, we can not determine $p \restriction n$.

Definition 8

Let $\Sigma = \{0, \pm 1\}$. The signed-digit representation ρ_{sd} is defined by

$$p \in \Sigma^\omega,\ \rho_{sd}(p) = \sum_{n=0}^{\infty} p(n)2^{-n-1} \in [-1,1].$$

The sd-representation can be extended to all reals.

Cylinder

For $\sigma \in \Sigma^{<\omega}$, let

$$[\sigma] = \{\rho(p) \ : \ \sigma \prec p \in \Sigma^{\omega}\},$$

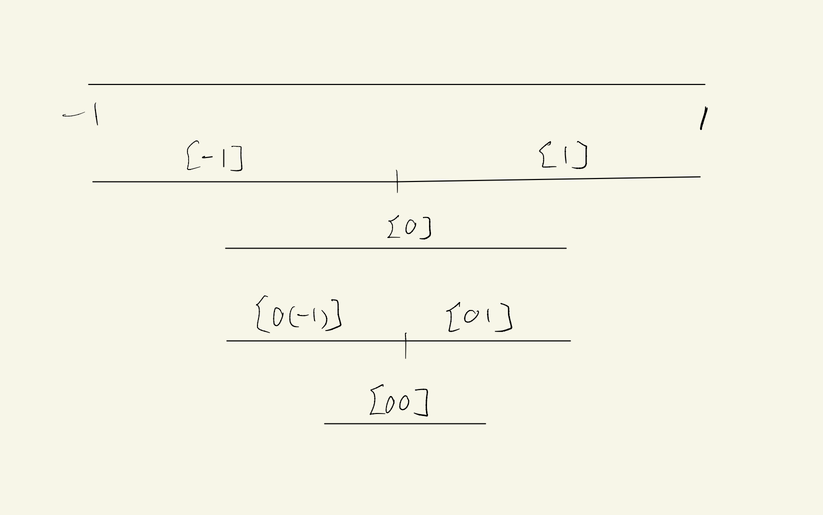
the set of the reals whose some sd-representation has an initial segment σ. Then, $[\sigma]$ is an interval $[a, b]$ with dyadic rationals with $|b - a| = 2^{-|\sigma|+1}$. Thus, fixing the initial segment with length n induces 2^{-n+1}-approximation.

We also have a converse. Let $I = [a, b]$ be some interval with $|I| \leq 2^{-n}$. Then, there exists $\sigma \in \Sigma^{<\omega}$ such that

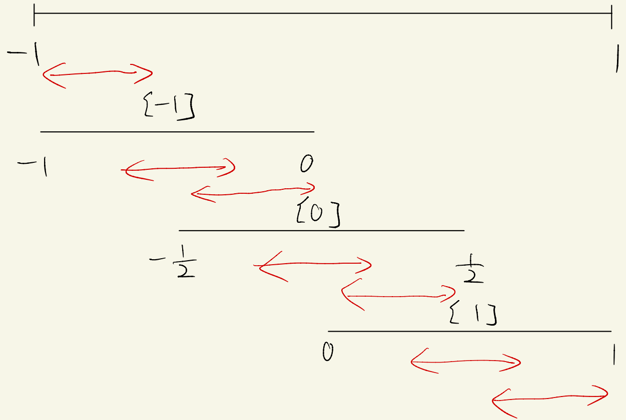
- $|\sigma| = n$,
- $I \subseteq [\sigma]$.

Thus, the interval with length $< 2^{-n}$ corresponds to the initial segment with length n.

Picture of Solovay reducibility



Example of cylinders



Realization

Definition 9

A partial computable $f :\subseteq \mathbb{R} \to \mathbb{R}$ is (ρ, ρ)-computable if $\exists \Phi :\subseteq \Sigma^\omega \to \Sigma^\omega$: Turing functional such that

$$(\forall p \in \Sigma^\omega)[\rho(p) = x \in \mathrm{dom}(f) \Rightarrow \rho(\Phi(p)) = f(x)].$$

We also say Φ realizes f.

Picture of realization

$$\begin{tikzcd}
x \in \mathbb{R} \arrow[rr, "f"] & & y = f(x) \in \mathbb{R} \\
p \in \Sigma^{\omega} \arrow[u, "\rho"] \arrow[rr, "\Phi"] & & \Phi(p) \in \Sigma^{\omega} \arrow[u]
\end{tikzcd}$$

Realization

Reproduce

Let $\alpha, \beta \in \mathbf{WC}$. $\alpha \leq_S \beta$ if and only if $\exists g$: partial comp. func. s.t.

- $\alpha = g(\beta)$,
- g is (ρ, ρ)-computable with use bound $H(n) = n + O(1)$,

where ρ is the signed-digit representation.

Here, g is defined at β but may not be defined at other reals.
For all ρ-representations B of β, Φ computes some ρ-representation A of α.
Furthermore, $A \upharpoonright n$ can be computed from $B \upharpoonright H(n)$.

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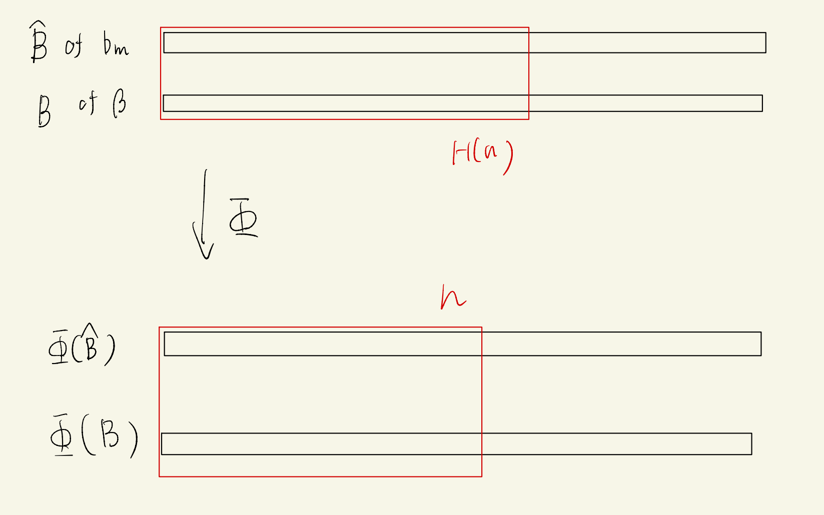
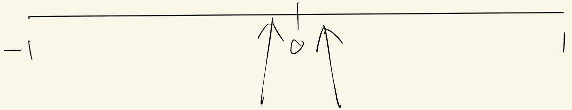
Proof idea

We give a proof sketch of one direction.
From a Turing functional Φ and $(b_n)_n$ converging to β, we construct $(a_n)_n$.
Fix a sufficiently large m.
Take some ρ-representation $\hat{B}$ of b_m.
If $\hat{B}$ and some ρ-representation B of β share initial $H(n)$-digits, then $\Phi(\hat{B})$ and $\Phi(B)$ share initial n-digits. Since $\Phi(B)$ is a ρ-representation of α, $\rho(\Phi(\hat{B}))$ is 2^{-n}-approximation of α.
However, even if b_m and β are close, this condition does not hold in general.
So we need to take some good ρ-representation $\hat{B}$ of b_m.

Picture of shareness



Bad representation



$\beta = -2^{-n}$ $\quad b_m = 2^{-n}$

very close!

Bad representation of b_m

$1(-1)(-1)(-1)(-1)\cdots\cdots \longrightarrow \beta \notin [1]$

This does not share initial segments with any ρ-rep of β.

Covering sequence

We can always retake a ρ representation such that the cylinders of their initial segments cover neighborhoods.

Proposition 10

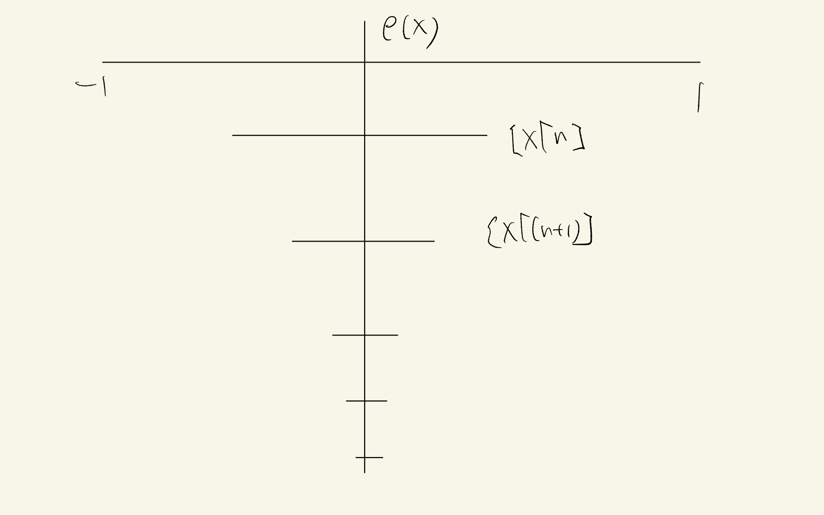
From a ρ-representation $X \in \Sigma^\omega$ of a real $x \in [-1, 1]$, one can compute another ρ-representation $X' \in \Sigma^\omega$ of the same real $x \in \mathbb{R}$ such that

$$[x - 2^{-n-3}, x + 2^{-n-3}] \cap [-1, 1] \subseteq [X' \upharpoonright n].$$

Furthermore, $X' \upharpoonright n$ depends only on $X \upharpoonright (n + 3)$.

By choosing such ρ-representation, the representations $\hat{B}$ of b_m and B of β share many initial segments if b_m and β are close.

Example of cylinders



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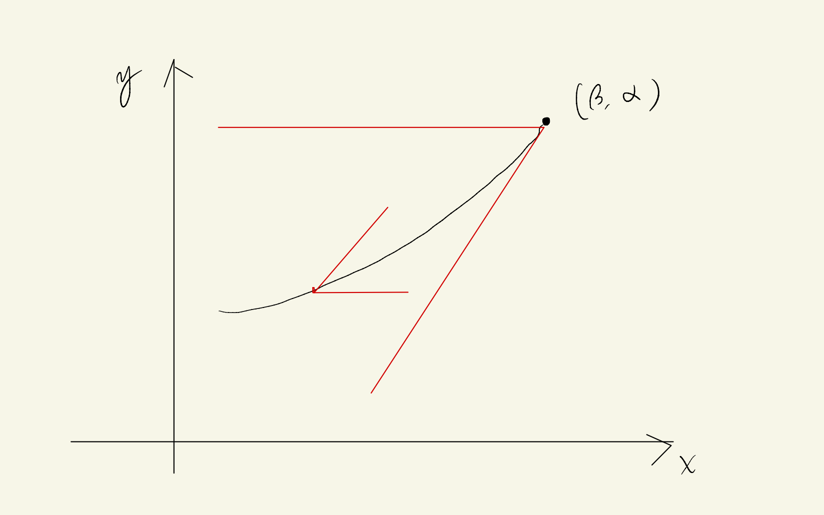
Solovay reducibility via Lipschitz functions

Proposition 11 (Kumabe, Miyabe, Mizusawa, and Suzuki 2020; Theorem 4.2)

Let α, β *be left-c.e. reals. Then* $\alpha \leq_S \beta$ *if and only if there exists a computable non-decreasing Lipschitz function* f *whose domain is* $(-\infty, \beta)$ *and* $\lim_{x \to \beta - 0} f(x) = \alpha$.

Solovay reducibility via Lipschitz functions



For weakly computable reals

Definition 12

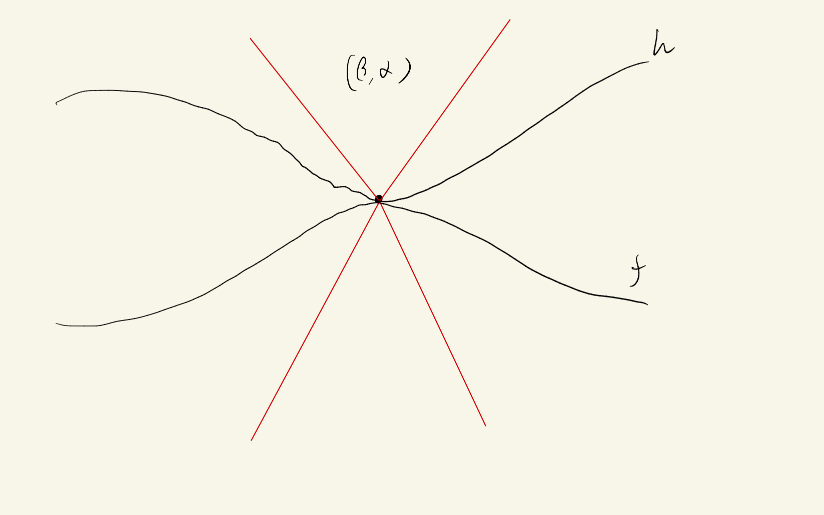
A function interval is the pair of two functions f and h with $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. A function interval (f, h) is semi-computable if f is lower semi-computable and h is upper semi-computable.

Theorem 13

Let $\alpha, \beta \in \mathbf{WC}$. Then, $\alpha \leq_S \beta$ if and only if there exist a semi-computable function interval (f, h) such that

1. *f, h are both Lipschitz functions,*
2. *$f(\beta) = h(\beta) = \alpha$.*

For weakly computable reals



Variants

An open interval $I = (a, b)$ is c.e. if a is a right-c.e. real and b is a left-c.e. real.

Definition 14 (cL-open reducibility)

For $\alpha, \beta \in \mathbb{R}$, α is computably-Lipschitz-reducible to β on a c.e. open interval, denoted by $\alpha \le_{cL}^{op} \beta$, if there exists a Lipschitz computable function f on a c.e. open interval I such that $\lim_{x \in I \to \beta} f(x) = \alpha$.

Definition 15 (cL-local reducibility)

For $\alpha, \beta \in \mathbb{R}$, α is computably-Lipschitz-reducible to β locally, denoted by $\alpha \le_{cL}^{loc} \beta$, if there exists a locally Lipschitz computable function f such that $f(\beta) = \alpha$.

Variants

Proposition 16

For weakly computable reals α, β, we have

$$\alpha \leq_{cL}^{loc} \beta \Rightarrow \alpha \leq_{cL}^{op} \beta \Rightarrow \alpha \leq_S \beta.$$

For left-c.e. reals α, β, $\alpha \leq_{cL}^{op} \beta$ if and only if $\alpha \leq_S \beta$.

Separation

Theorem 17

There exist left-c.e. reals α, β *such that* $\alpha \leq_{cL}^{op} \beta$ *but* $\alpha \not\leq_{cL}^{loc} \beta$.

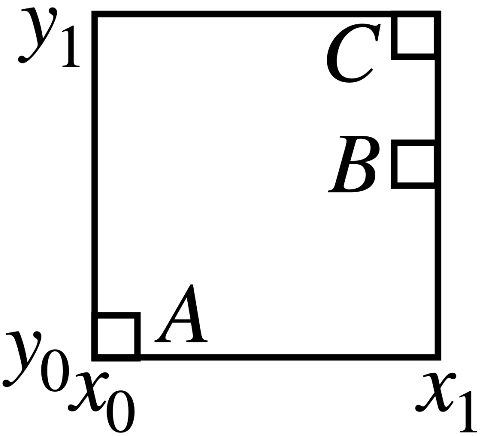
Theorem 18

There exist $\alpha, \beta \in \mathbf{WC}$ *such that* $\alpha \leq_S \beta$ *but* $\alpha \not\leq_{cL}^{op} \beta$.

Proof sketch

- We enumerate partial computable functions that contain all total Lipschitz functions.
- For each function f, by fixing initial segments of α, β, we assure that $\alpha \not\leq_{cL}^{loc} \beta$ via this f.
- We change approximations of α, β at the same time so that $\alpha \leq_S \beta$.

Separation

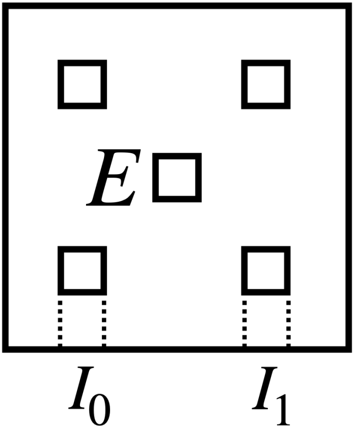


Proof sketch

- We have already fixed initial segments so that (β, α) is in this square.
- First, set $(\beta, \alpha) \in A$.
- Compute $f(x_1)$. This may be undefined. In that case, we continue to stay in A.
- If $f(x_1) > y'$, then move to B. If $f(x_1) < y'$, then move to C. Here, $y' \approx \frac{y_0 + 3y_1}{4}$.

If $f(\beta) = \alpha$, f should be a steep slope.
We continue this strategy for the next f in the smaller squares A, B, or C.
Requirements with high priority may injure the requirements with low priority at most finite times.

Separation



Further results

- When replacing Lipschitz by Hölder, then we have quasi Solovay reducibility, where the use bound is $H(n) = pn + O(1)$ for some $p \in \omega$.
- When defining strong Slovay reducibility by $\lim_n \frac{\alpha - a_n}{\beta - b_n} = 0$, then we have
  - The derivative of f at β is 0.
  - The use bound $H(n) < n - d$ for any d.

Question

Question 19

Are there left-c.e. ML-random reals α, β such that $\alpha \leq_{cL}^{loc} \beta$?

We have already shown the existence when dropping ML-randomness.
This seems interesting because

1. the computable Lipschitz function converging α when $x \to \beta - 0$ have positive left-derivative $\lim_s \frac{\alpha - a_n}{\beta - b_n}$ by Barmpalias and Lewis-Pye's result.
2. any computable Lipschitz function is differentiable at any computable random point by Brattka, Miller, and Nies's result.

Thank you!