Solovay reducibility and signed-digit representation

Kenshi Miyabe
Meiji University
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Joint work with Masahiro Kumabe (Open Univ.) and Toshio Suzuki (Tokyo Metropolitan Univ.).
We give some new characterizations of Solovay reducibility for weakly computable reals.

1. By Turing reduction with bounded use with respect to the signed-digit representation.

2. By upper and lower semi-computable Lipschitz functions.

“Solovay reducibility” is well-behaved even outside of left-c.e. reals!!
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Main result

Computability of reals

\( \alpha \in \mathbb{R} \) \text{ is computable } \text{ if } \exists (a_n)_n, \text{ comp, } |a_n - \alpha| < 2^{-n} \text{ for all } n \in \omega. \\
\alpha \in \mathbb{R} \text{ is left-c.e. } \text{ if } \exists (a_n)_n, \text{ comp., increasing, converging to } \alpha \\
\alpha \in \mathbb{R} \text{ is weakly computable (d.c.e., d.l.c.e.)} \\
\text{ if } \exists (a_n)_n, \text{ comp. } \sum_n |a_{n+1} - a_n| < \infty, \text{ converging to } \alpha, \\
\text{ or equivalently if } \alpha = \beta - \gamma \text{ for left-c.e. reals } \beta, \gamma.

\text{EC } \subsetneq \text{ LC } \subsetneq \text{ WC.}
Main result

Picture of computability

\[ \text{computable} \rightarrow \text{left-c.e.} \rightarrow \text{weakly computable} \]
Main result

Solovay reducibility for left-c.e. reals

Definition 1 (Solovay 1970s)
Let $\alpha, \beta \in \text{LC}$. $\alpha$ is Solovay reducible to $\beta$, denoted by $\alpha \leq_S \beta$, if $\exists (a_n)_n, (b_n)_n$, comp., increasing, converging to $\alpha, \beta$ and $\exists c \in \omega$ such that

$$\alpha - a_n < c(\beta - b_n).$$

If one has a good approximation of $\beta$ from below, then one can compute a good approximation of $\alpha$.

Theorem 2 (Kučera and Slaman 2001 with some other results)
A left-c.e. real $\beta$ is Martin-Löf random if and only if it is Solovay complete in left-c.e. reals, that is, $\alpha \leq_S \beta$ for all left-c.e. reals $\alpha$. 
Main result

Picture of Solovay reducibility
Main result

Solovay reducibility for weakly computable reals

Definition 3 (Zheng and Rettinger 2004)

Let \( \alpha, \beta \in WC \). \( \alpha \) is **Solovay reducible** to \( \beta \), denoted by \( \alpha \leq_S \beta \), if there exist sequences \((a_n)_n, (b_n)_n\), computable, converging to \( \alpha, \beta \) and \( c \in \omega \) such that

\[
|\alpha - a_n| < c(|\beta - b_n| + 2^{-n}).
\]

\((a_n)_n, (b_n)_n\) need not be increasing.

The definition also works for limit computable reals (=computably approximable reals), but we focus on weakly computable reals for simplicity.

Proposition 4 (Rettinger and Zheng 2005)

*If a weakly computable real is ML-random, then it is left-c.e. or right-c.e.*

A weakly computable real \( \beta \) is **Martin-Löf random** if and only if it is Solovay complete in weakly computable reals.

Thus, Solovay is complete if and only if left-c.e. ML-random (\( \Omega \)) or right-c.e. ML-random(−\( \Omega \)).
Main result

Picture of Solovay reducibility
Main result

Picture of Solovay reducibility

\[ \text{Top} \]
\[ \text{weakly comp. reals} \]
\[ = \Omega \text{ or } -\Omega \]

\[ \text{ML-random} \]
\[ \text{comp.} \]

\[ \text{bottom} \]
Question

The original Solovay reducibility is well-behaved within left-c.e. reals. The Solovay reducibility by Zheng and Rettinger is well-behaved within weakly computable reals.

Question 5

Is there a reducibility such that

1. it has many good properties like Solovay reducibility,
2. it is well-behaved for all reals.
**cL-reducibility**

It would be desirable that Solovay reducibility can be characterized via Turing use bounds like tt and wtt.

**Definition 6 (Downey, Hirschfeldt, and LaForte 2004)**

Let $\alpha, \beta \in 2^\omega$. Then, $\alpha$ is computably Lipschitz reducible to $\beta$, denoted by $\alpha \leq_{cL} \beta$, if $\exists \Phi$: Turing functional s.t.

- $\alpha = \Phi(\beta)$,
- $\text{use}(\Phi, \beta, n) \leq n + O(1)$.

Solovay reducibility requires us to compute $2^{-n}$-approximation of $\alpha$ from $2^{-n-O(1)}$-approximation of $\beta$. In this sense, these reducibilities are similar but, unfortunately, incomparable (see Theorem 9.1.6 and 9.10.1 in Downey and Hirschfeldt 2010).
Main result

Picture of Solovay reducibility

\[ h \in O(1) \]
The main reason for the difference between Solovay reducibility and cL-reducibility is that the reals change continuously, while the binary sequences change discretely. A similar problem occurs in computable analysis, where we use the signed-digit representation for reals.

**Theorem 7 (Kumabe, M., Suzuki)**

Let \( \alpha, \beta \in WC \). \( \alpha \leq_S \beta \) if and only if it \( \exists g: \) partial comp. func. s.t.

1. \( \alpha = g(\beta) \),
2. \( g \) is \((\rho, \rho)\)-computable with use bound \( H(n) = n + O(1) \),

where \( \rho \) is the signed-digit representation.

We will define the sd-representation later. Replacing the binary representation in cL-reducibility with the sd-representation characterizes Solovay reducibility!
Main result

Solovay reducibility for all reals

This feature is pleasing in several ways.

- We have Solovay reducibility for all reals by redefining it via the use bound w.r.t. sd-representation.
- The condition uses use bound like many other reducibilities in computability theory.
- This clarifies the relation between Solovay reducibility and Lipschitz functions.
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Definition of sd-representation

The usual binary representation:

\[ p \in 2^\omega, \quad \rho_{bin}(p) = \sum_{n=0}^{\infty} p(n)2^{-n-1} \in [0, 1]. \]

Even if \( \alpha \in [a, b] \) with \( b - a < 2^{-n} \), we can not determine \( p \upharpoonright n \).

**Definition 8**

Let \( \Sigma = \{0, \pm 1\} \). The signed-digit representation \( \rho_{sd} \) is defined by

\[ p \in \Sigma^\omega, \quad \rho_{sd}(p) = \sum_{n=0}^{\infty} p(n)2^{-n-1} \in [-1, 1]. \]

The sd-representation can be extended to all reals.
Cylinder

For \( \sigma \in \Sigma^{<\omega} \), let

\[
[\sigma] = \{ \rho(p) : \sigma \prec p \in \Sigma^\omega \},
\]

the set of the reals whose some sd-representation has an initial segment \( \sigma \). Then, \([\sigma]\) is an interval \([a, b]\) with dyadic rationals with \(|b - a| = 2^{-|\sigma|+1}\). Thus, fixing the initial segment with length \( n \) induces \(2^{-n+1}\)-approximation.

We also have a converse. Let \( I = [a, b] \) be some interval with \(|I| \leq 2^{-n}\). Then, there exists \( \sigma \in \Sigma^{<\omega} \) such that

- \(|\sigma| = n,\)
- \(I \subseteq [\sigma].\)

Thus, the interval with length \(< 2^{-n}\) corresponds to the initial segment with length \( n \).
Picture of Solovay reducibility
Example of cylinders

Signed-digit representation

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Realization

Definition 9

A partial computable \( f : \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is \((\rho, \rho)\)-computable if \( \exists \Phi : \subseteq \Sigma^\omega \rightarrow \Sigma^\omega : \) Turing functional such that

\[
(\forall p \in \Sigma^\omega)[\rho(p) = x \in \text{dom}(f) \Rightarrow \rho(\Phi(p)) = f(x)].
\]

We also say \( \Phi \) realizes \( f \).
Signed-digit representation

Picture of realization

\[ x \in \mathbb{R} \xrightarrow{f} y = f(x) \in \mathbb{R} \]

\[ p \in \Sigma^{\omega} \xrightarrow{\Phi} \Phi(p) \in \Sigma^{\omega} \]
Reproduce

Let $\alpha, \beta \in \mathbf{WC}$. $\alpha \leq_S \beta$ if and only if $\exists g$: partial comp. func. s.t.
- $\alpha = g(\beta)$,
- $g$ is $(\rho, \rho)$-computable with use bound $H(n) = n + O(1)$,

where $\rho$ is the signed-digit representation.

Here, $g$ is defined at $\beta$ but may not be defined at other reals.
For all $\rho$-representations $B$ of $\beta$, $\Phi$ computes some $\rho$-representation $A$ of $\alpha$.
Furthermore, $A \upharpoonright n$ can be computed from $B \upharpoonright H(n)$.
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Proof idea

We give a proof sketch of one direction. From a Turing functional \( \Phi \) and \((b_n)_n\) converging to \( \beta \), we construct \((a_n)_n\). Fix a sufficiently large \( m \). Take some \( \rho \)-representation \( \hat{B} \) of \( b_m \).

If \( \hat{B} \) and some \( \rho \)-representation \( B \) of \( \beta \) share initial \( H(n) \)-digits, then \( \Phi(\hat{B}) \) and \( \Phi(B) \) share initial \( n \)-digits. Since \( \Phi(B) \) is a \( \rho \)-representation of \( \alpha \), \( \rho(\Phi(\hat{B})) \) is \( 2^{-n} \)-approximation of \( \alpha \).

However, even if \( b_m \) and \( \beta \) are close, this condition does not hold in general. So we need to take some good \( \rho \)-representation \( \hat{B} \) of \( b_m \).
Proof

Picture of shareness

\[ \hat{B} \text{ of } b_m \]
\[ B \text{ of } \beta \]
\[ H(a) \]
\[ \Phi \]
\[ \Phi(B) \]
\[ \Phi(B) \]
\[ \Phi \]
\[ h(n) \]
\[ H(\lambda) \]

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Proof

Bad representation

\[ \beta = -2^{-n} \quad b_m = 2^{-n} \]

very close!

Bad representation of \( b_m \)

\[ (-1)(-1)(-1)(-1) \cdots \rightarrow \beta \in [1] \]

This does not share initial segments with any \( \rho \)-rep of \( \beta \).
Proof

Covering sequence

We can always retake a $\rho$ representation such that the cylinders of their initial segments cover neighborhoods.

**Proposition 10**

*From a $\rho$-representation $X \in \Sigma^\omega$ of a real $x \in [-1, 1]$, one can compute another $\rho$-representation $X' \in \Sigma^\omega$ of the same real $x \in \mathbb{R}$ such that*

$$[x - 2^{-n-3}, x + 2^{-n-3}] \cap [-1, 1] \subseteq [X' \upharpoonright n].$$

*Furthermore, $X' \upharpoonright n$ depends only on $X \upharpoonright (n + 3)$.*

By choosing such $\rho$-representation, the representations $\hat{B}$ of $b_m$ and $B$ of $\beta$ share many initial segments if $b_m$ and $\beta$ are close.
Proof

Example of cylinders
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Proposition 11 (Kumabe, Miyabe, Mizusawa, and Suzuki 2020; Theorem 4.2)

Let $\alpha, \beta$ be left-c.e. reals. Then $\alpha \leq_S \beta$ if and only if there exists a computable non-decreasing Lipschitz function $f$ whose domain is $(-\infty, \beta)$ and

$$\lim_{x \to \beta^-} f(x) = \alpha.$$
Solovay reducibility via Lipschitz functions
For weakly computable reals

**Definition 12**

A function interval is the pair of two functions $f$ and $h$ with $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. A function interval $(f, h)$ is semi-computable if $f$ is lower semi-computable and $h$ is upper semi-computable.

**Theorem 13**

Let $\alpha, \beta \in WC$. Then, $\alpha \leq_S \beta$ if and only if there exist a semi-computable function interval $(f, h)$ such that

1. $f, h$ are both Lipschitz functions,
2. $f(\beta) = h(\beta) = \alpha$. 
For weakly computable reals
An open interval $I = (a, b)$ is c.e. if $a$ is a right-c.e. real and $b$ is a left-c.e. real.

**Definition 14 (cL-open reducibility)**

For $\alpha, \beta \in \mathbb{R}$, $\alpha$ is computably-Lipschitz-reducible to $\beta$ on a c.e. open interval, denoted by $\alpha \leq_{\text{cL}}^{\text{op}} \beta$, if there exists a Lipschitz computable function $f$ on a c.e. open interval $I$ such that $\lim_{x \in I \to \beta} f(x) = \alpha$.

**Definition 15 (cL-local reducibility)**

For $\alpha, \beta \in \mathbb{R}$, $\alpha$ is computably-Lipschitz-reducible to $\beta$ locally, denoted by $\alpha \leq_{\text{cL}}^{\text{loc}} \beta$, if there exists a locally Lipschitz computable function $f$ such that $f(\beta) = \alpha$. 
Proposition 16

*For weakly computable reals* $\alpha, \beta$, we have

$$\alpha \leq_{loc} \beta \Rightarrow \alpha \leq_{op} \beta \Rightarrow \alpha \leq_S \beta.$$ 

For left-c.e. reals $\alpha, \beta$, $\alpha \leq_{op} \beta$ if and only if $\alpha \leq_S \beta$. 
Separation

Theorem 17

There exist left-c.e. reals $\alpha, \beta$ such that $\alpha \leq_{cL}^\text{op} \beta$ but $\alpha \not\leq_{cL}^\text{loc} \beta$.

Theorem 18

There exist $\alpha, \beta \in WC$ such that $\alpha \leq_S \beta$ but $\alpha \not\leq_{cL}^\text{op} \beta$. 
Proof sketch

- We enumerate partial computable functions that contain all total Lipschitz functions.
- For each function $f$, by fixing initial segments of $\alpha, \beta$, we assure that $\alpha \not\leq^{loc}_{cL} \beta$ via this $f$.
- We change approximations of $\alpha, \beta$ at the same time so that $\alpha \leq_{S} \beta$. 
Separation

\[ y_1 \]

\[ y_0 \]

\[ x_0 \]

\[ x_1 \]
Proof sketch

- We have already fixed initial segments so that $(\beta, \alpha)$ is in this square.
- First, set $(\beta, \alpha) \in A$.
- Compute $f(x_1)$. This may be undefined. In that case, we continue to stay in $A$.
- If $f(x_1) > y'$, then move to $B$. If $f(x_1) < y'$, then move to $C$. Here, $y' \approx \frac{y_0 + 3y_1}{4}$.

If $f(\beta) = \alpha$, $f$ should be a steep slope.
We continue this strategy for the next $f$ in the smaller squares $A$, $B$, or $C$.
Requirements with high priority may injure the requirements with low priority at most finite times.
Other results

Separation

\[ \begin{array}{c}
E \\
I_0 \\
I_1 \\
\end{array} \]
Further results

- When replacing Lipschitz by Hölder, then we have quasi Solovay reducibility, where the use bound is $H(n) = pn + O(1)$ for some $p \in \omega$.
- When defining strong Slovay reducibility by $\lim_{n} \frac{\alpha - a_n}{\beta - b_n} = 0$, then we have
  - The derivative of $f$ at $\beta$ is 0.
  - The use bound $H(n) < n - d$ for any $d$. 
Are there left-c.e. ML-random reals $\alpha, \beta$ such that $\alpha \leq_{cL}^{loc} \beta$?

We have already shown the existence when dropping ML-randomness. This seems interesting because

1. the computable Lipschitz function converging $\alpha$ when $x \to \beta - 0$ have positive left-derivative $\lim_s \frac{\alpha - a_n}{\beta - b_n}$ by Barmpalias and Lewis-Pye’s result.

2. any computable Lipschitz function is differentiable at any computable random point by Brattka, Miller, and Nies’s result.

Thank you!