Solovay reducibility and signed-digit representation

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Joint work with Masahiro Kumabe (Open Univ.) and Toshio Suzuki (Tokyo Metropolitan Univ.).

I talked about the same topic at NUS last June. I will focus on a different proof this time.

Goal

We give some new characterizations of Solovay reducibility for weakly computable reals.

1. By upper and lower semi-computable Lipschitz functions.
2. By Turing reduction with bounded use with respect to the signed-digit representation.

"Solovay reducibility" is well-behaved even outside of left-c.e. reals!!

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Computability of reals

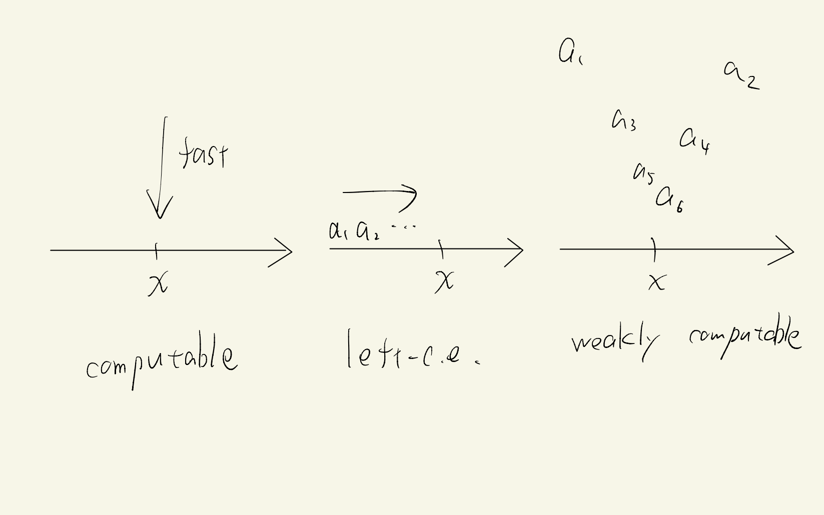
$\alpha \in \mathbb{R}$ is computable if $\exists (a_n)_n$, comp, $|a_n - \alpha| < 2^{-n}$ for all $n \in \omega$.

$\alpha \in \mathbb{R}$ is left-c.e. if $\exists (a_n)_n$, comp., increasing, converging to α

$\alpha \in \mathbb{R}$ is weakly computable (d.c.e., d.l.c.e.)
if $\exists (a_n)_n$, comp. $\sum_n |a_{n+1} - a_n| < \infty$, converging to α,
or equivalently if $\alpha = \beta - \gamma$ for left-c.e. reals β, γ.

$$\mathbf{EC} \subsetneq \mathbf{LC} \subsetneq \mathbf{WC}.$$

Picture of computability



Solovay reducibility for left-c.e. reals

Definition 1 (Solovay 1970s)

Let $\alpha, \beta \in \mathbf{LC}$. α is Solovay reducible to β, denoted by $\alpha \leq_S \beta$, if $\exists (a_n)_n, (b_n)_n$, comp., increasing, converging to α, β and $\exists c \in \omega$ such that

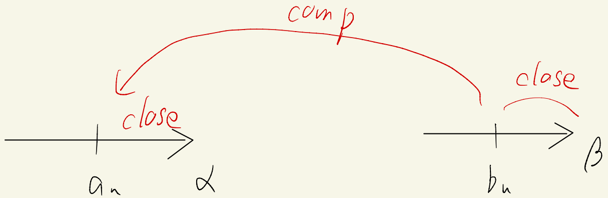
$$\alpha - a_n < c(\beta - b_n).$$

If one has a good approximation of β from below, then one can compute a good approximation of α.

Theorem 2 (Kučera and Slaman 2001 with some other results)

A left-c.e. real β is Martin-Löf random if and only if it is Solovay complete in left-c.e. reals, that is, $\alpha \leq_S \beta$ for all left-c.e. reals α.

Picture of Solovay reducibility



Solovay reducibility for weakly computable reals

Definition 3 (Zheng and Rettinger 2004)

Let $\alpha, \beta \in \mathbf{WC}$. α is Solovay reducible to β, denoted by $\alpha \leq_S \beta$, if $\exists (a_n)_n, (b_n)_n$, comp., converging to α, β and $\exists c \in \omega$ such that

$$|\alpha - a_n| < c(|\beta - b_n| + 2^{-n}).$$

$(a_n)_n, (b_n)_n$ need not be increasing

The definition also works for limit computable reals (=computably approximable reals), but we focus on weakly computable reals for simplicity.

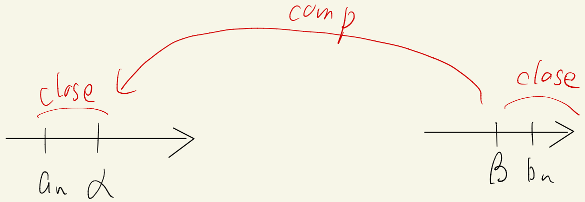
Proposition 4 (Rettinger and Zheng 2005)

If a weakly computable real is ML-random, then it is left-c.e. or right-c.e.

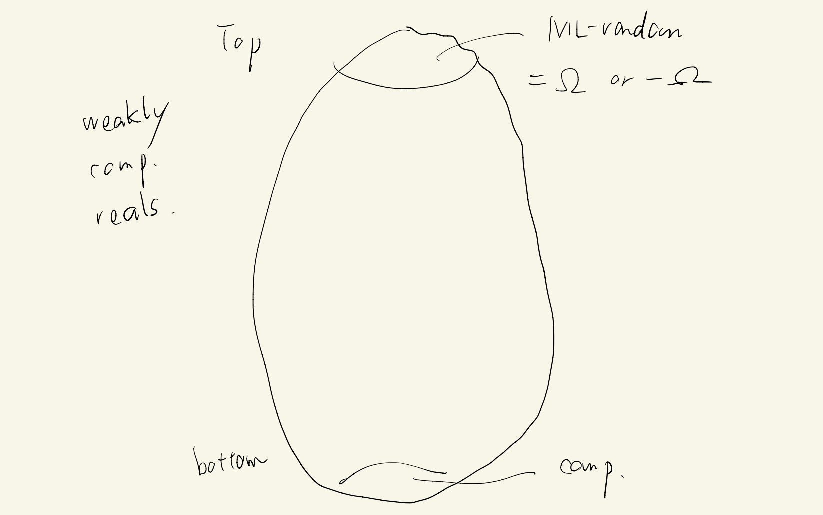
A weakly computable real β is Martin-Löf random if and only if it is Solovay complete in weakly computable reals.

Thus, Solovay is complete if and only if left-c.e. ML-random (Ω) or right-c.e. ML-random($-\Omega$).

Picture of Solovay reducibility



Picture of Solovay reducibility



Question

Question 5

What does it mean for one real number to be more random than another number?

The original Solovay reducibility is well-behaved within left-c.e. reals.

The Solovay reducibility by Zheng and Rettinger is well-behaved within weakly computable reals.

Is there a reducibility such that

1. it has many good properties like Solovay reducibility,
2. it is well-behaved for all reals.

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Solovay reducibility via Lipschitz functions

$f : \mathbb{R} \to \mathbb{R}$ is Lipschitz if there exists a constant $L \in \mathbb{R}$ such that

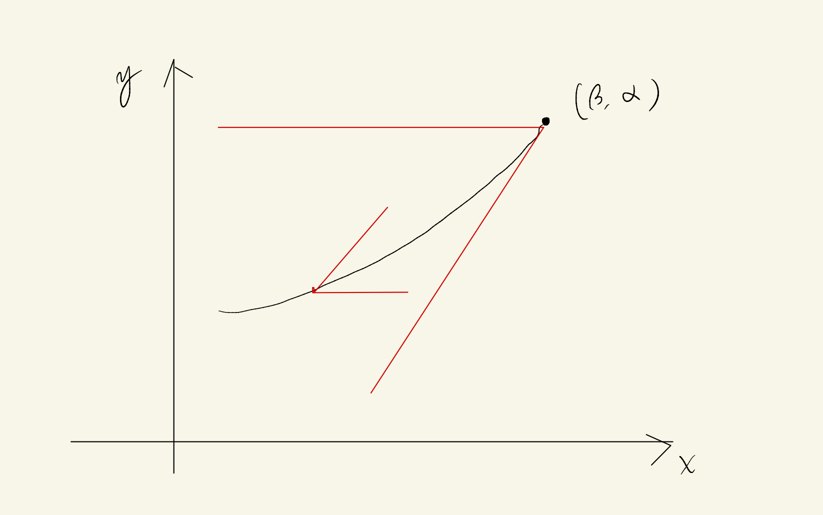
$$|f(x) - f(y)| \leq L|x - y|$$

for all x, y. If f is C^1 in a closed interval, then it is Lipschitz.

Proposition 6 (Kumabe, Miyabe, Mizusawa, and Suzuki 2020; Theorem 4.2)

Let α, β be left-c.e. reals. Then $\alpha \leq_S \beta$ if and only if there exists a computable non-decreasing Lipschitz function f whose domain is $(-\infty, \beta)$ and $\lim_{x \to \beta - 0} f(x) = \alpha$.

Solovay reducibility via Lipschitz functions



For weakly computable reals

Definition 7

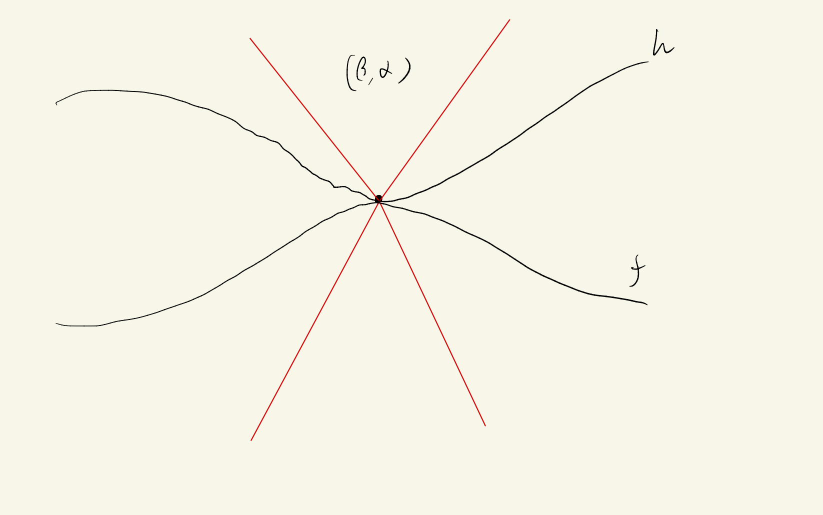
A function interval is the pair of two functions f and h with $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. A function interval (f, h) is semi-computable if f is lower semi-computable and h is upper semi-computable.

Theorem 8

Let $\alpha, \beta \in \mathbf{WC}$. Then, $\alpha \leq_S \beta$ if and only if there exist a semi-computable function interval (f, h) such that

1. *f, h are both Lipschitz functions,*
2. *$f(\beta) = h(\beta) = \alpha$.*

For weakly computable reals



Cauchy-type characterization

For left-c.e. reals α, β, $\alpha \leq_S \beta$ if and only if there exists non-decreasing computable sequences $(a_n)_n$ and $(b_n)_n$ converging to α and β, respectively, and $q \in \omega$ such that

$$(\forall n) a_{n+1} - a_n < q(b_{n+1} - b_n).$$

Proposition 9

For weakly computable reals α, β, the relation $\alpha \leq_S \beta$ holds if and only if there exist computable sequences $(a_n)_n$ and $(b_n)_n$ converging to α and β respectively and $q \in \omega$ such that

$$(\forall k, n \in \omega)[k < n \Rightarrow |a_n - a_k| < q \cdot (|b_n - b_k| + 2^{-k})]$$

We do not know any adjacent version of Solovay reducibility for weakly computable reals.

Proof idea

The “if” direction follows by letting $n \to \infty$.
For the “only if” direction, we can always find such subsequences.
Suppose that we have

$$|\alpha - a_k| < c(|\beta - b_k| + 2^{-k}).$$

For all sufficiently large n, the desired property holds for all previous terms.

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \qquad \alpha$

$b_1 \quad b_2 \quad b_3 \quad b_4 \quad b_5 \qquad \beta$

$c_1 \quad c_2 \quad c_3$

$d_1 \quad d_2 \quad d_3$

$$|\alpha - a_k| < C(|\beta - b_k| + 2^{-k})$$

Proof idea

The Cauchy-type characterization states that

$$(\forall k, n)[k < n \Rightarrow a_k - q|b_n - b_k| - 2^{-k} < a_n < a_k + q|b_n - b_k| + 2^{-k}].$$

Inspired from this, we define functions f and h as follows:

(a) $f(x) = \sup_{n\in\omega}(a_n - q|x - b_n| - 2^{-n})$,

(b) $h(x) = \inf_{n\in\omega}(a_n + q|x - b_n| + 2^{-n})$.

Then, we can show that

- f is lower semi-computable and h is upper semi-computable,
- $f(x) \leq h(x)$ for all $x \in \mathbb{R}$,
- f, h are both Lipschitz functions,
- $f(\beta) = h(\beta) = \alpha$.

Variants

An open interval $I = (a, b)$ is c.e. if a is a right-c.e. real and b is a left-c.e. real.

Definition 10 (cL-open reducibility)

For $\alpha, \beta \in \mathbb{R}$, α is computably-Lipschitz-reducible to β on a c.e. open interval, denoted by $\alpha \leq_{cL}^{op} \beta$, if there exists a Lipschitz computable function f on a c.e. open interval I such that $\lim_{x \in I \to \beta} f(x) = \alpha$.

Definition 11 (cL-local reducibility)

For $\alpha, \beta \in \mathbb{R}$, α is computably-Lipschitz-reducible to β locally, denoted by $\alpha \leq_{cL}^{loc} \beta$, if there exists a locally Lipschitz computable function f such that $f(\beta) = \alpha$.

Variants

Proposition 12

For weakly computable reals α, β, we have

$$\alpha \leq_{cL}^{loc} \beta \Rightarrow \alpha \leq_{cL}^{op} \beta \Rightarrow \alpha \leq_S \beta.$$

For left-c.e. reals α, β, $\alpha \leq_{cL}^{op} \beta$ if and only if $\alpha \leq_S \beta$.

Separation

Theorem 13

There exist left-c.e. reals α, β *such that* $\alpha \leq_{cL}^{op} \beta$ *but* $\alpha \not\leq_{cL}^{loc} \beta$.

Theorem 14

There exist $\alpha, \beta \in \mathbf{WC}$ *such that* $\alpha \leq_S \beta$ *but* $\alpha \not\leq_{cL}^{op} \beta$.

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cL-reducibility

It would be desirable that Solovay reducibility can be characterized via Turing use bounds like tt and wtt.

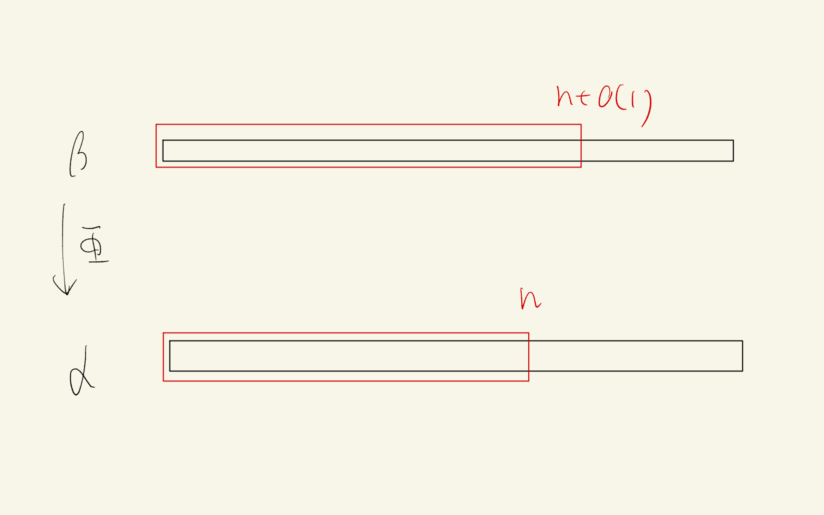
Definition 15 (Downey, Hirschfeldt, and LaForte 2004)

Let $\alpha, \beta \in 2^\omega$. Then, α is computably Lipschitz reducible to β, denoted by $\alpha \leq_{cL} \beta$, if $\exists \Phi$: Turing functional s.t.

- $\alpha = \Phi(\beta)$,
- $\mathrm{use}(\Phi, \beta, n) \leq n + O(1)$.

Solovay reducibility requires us to compute 2^{-n}-approximation of α from $2^{-n-O(1)}$-approximation of β. In this sense, these reducibilities are similar but, unfortunately, incomparable (see Theorem 9.1.6 and 9.10.1 in Downey and Hirschfeldt 2010).

Picture of Solovay reducibility



signed-digit representation

The main reason for the difference between Solovay reducibility and cL-reducibility is that the reals change continuously, while the binary sequences change discretely. A similar problem occurs in computable analysis, where we use the signed-digit representation for reals.

Theorem 16 (Kumabe, M., Suzuki)

Let $\alpha, \beta \in \mathbf{WC}$. $\alpha \leq_S \beta$ *if and only if it* $\exists g$*: partial comp. func. s.t.*

- $\alpha = g(\beta)$,
- g *is* (ρ, ρ)*-computable with use bound* $H(n) = n + O(1)$,

where ρ *is the signed-digit representation.*

We will define the sd-representation later.
Replacing the binary representation in cL-reducibility with the sd-representation characterizes Solovay reducibility!

Solovay reducibility for all reals

This feature is pleasing in several ways.

- We have Solovay reducibility for all reals by redefining it via the use bound w.r.t. sd-representation.
- The condition uses use bound like many other reducibilities in computability theory.
- This clarifies the relation between Solovay reducibility and Lipschitz functions.

Definition of sd-representation

The usual binary representation:

$$p \in 2^\omega, \ \rho_{bin}(p) = \sum_{n=0}^{\infty} p(n) 2^{-n-1} \in [0,1].$$

Even if $\alpha \in [a,b]$ with $b - a < 2^{-n}$, we can not determine $p \upharpoonright n$.

Definition 17

Let $\Sigma = \{0, \pm 1\}$. The signed-digit representation ρ_{sd} is defined by

$$p \in \Sigma^\omega, \ \rho_{sd}(p) = \sum_{n=0}^{\infty} p(n) 2^{-n-1} \in [-1,1].$$

The sd-representation can be extended to all reals.

Realization

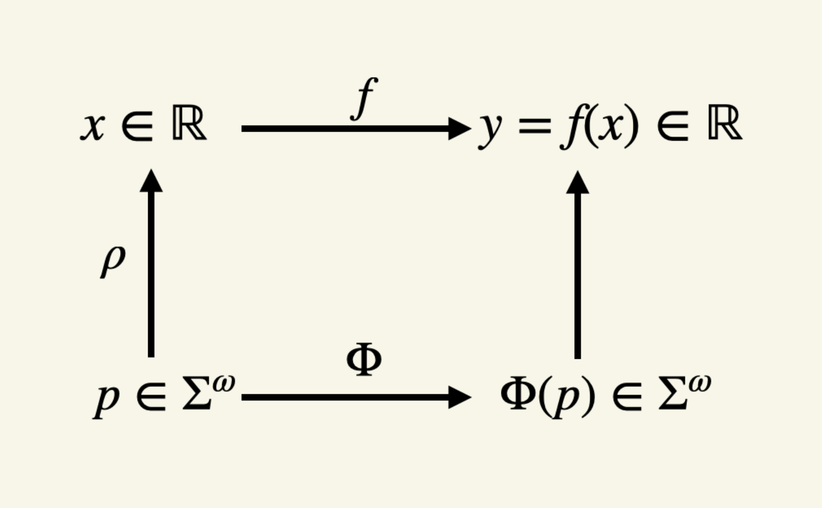
Definition 18

A partial computable $f :\subseteq \mathbb{R} \to \mathbb{R}$ is (ρ, ρ)-computable if $\exists \Phi :\subseteq \Sigma^\omega \to \Sigma^\omega$: Turing functional such that

$$(\forall p \in \Sigma^\omega)[\rho(p) = x \in \mathrm{dom}(f) \Rightarrow \rho(\Phi(p)) = f(x)].$$

We also say Φ realizes f.

Picture of realization



Realization

Reproduce

Let $\alpha, \beta \in \mathbf{WC}$. $\alpha \leq_S \beta$ if and only if $\exists g$: partial comp. func. s.t.

- $\alpha = g(\beta)$,
- g is (ρ, ρ)-computable with use bound $H(n) = n + O(1)$,

where ρ is the signed-digit representation.

Here, g is defined at β but may not be defined at other reals.
For all ρ-representations B of β, Φ computes some ρ-representation A of α.
Furthermore, $A \upharpoonright n$ can be computed from $B \upharpoonright H(n)$.

Further results

- When replacing Lipschitz by Hölder, then we have quasi Solovay reducibility, where the use bound is $H(n) = pn + O(1)$ for some $p \in \omega$.
- When defining strong Slovay reducibility by $\lim_n \frac{\alpha - a_n}{\beta - b_n} = 0$, then we have
  - The derivative of f at β is 0.
  - The use bound $H(n) < n - d$ for any d.

Thank you!