SOLOVAY REDUCIBILITY VIA LIPSCHITZ FUNCTIONS AND SIGNED-DIGIT REPRESENTATION

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Abstract. We explore Solovay reducibility in the context of computably approximable reals, extending its natural characterization for left-c.e. reals via computable Lipschitz functions. Our paper offers two distinct characterizations: the first employs Lipschitz functions, while the second utilizes Turing reductions with bounded use with respect to signed-digit representation. Additionally, we examine multiple related reducibilities and establish separations among them. These results contribute to a refined perspective of the relationship between Solovay reducibility and computable Lipschitz functions.

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1. Introduction

1.1. **Background.** The theory of algorithmic randomness [3, 11] specifies what it means for a real number to be random. The most studied concept of randomness is probably Martin-Löf randomness, which has many good properties as a randomness notion. See [9, Section 3.10] for a history of this topic.

The next problem is to define which of the two real numbers is more random than the other. There are many reducibilities defined in the literature. One such reducibility is Solovay reducibility. Informally, one real is Solovay reducible to another if one can construct a good approximation of the former from any good approximation of the latter. Although Solovay reducibility has many advantages, it behaves well only for left-c.e. reals.

A real is called left-c.e. if it has a computable approximation from below. The left-c.e. reals are well-studied in the theory of algorithmic randomness, but the set of all left-c.e. reals is not closed under even subtraction. The set of all weakly computable reals is a more natural class of reals. For example, the set of all weakly computable reals forms a real closed field (Ng [10] and Raichev [12]).

Zheng and Rettinger [18] defined a reducibility that coincides with Solovay reducibility for left-c.e. reals and well-behaves even outside left-c.e. reals. In this paper, we refer to this reducibility simply as Solovay reducibility.

1.2. **Characterization via Lipschitz functions.** Relatively recently, Kumabe, Miyabe, Mizusawa, and Suzuki [7] noticed that Solovay reducibility for left-c.e. reals has a natural characterization via computable Lipschitz functions. An approximation of α may change significantly only when that of β does as well. Such boundedness of the slope from above corresponds to the Lipschitz condition.

The first question discussed in this paper is whether Solovay reducibility for computably approximable reals is characterized by Lipschitz functions. We affirm this in Theorem 3.8. While Solovay reducibility for left-c.e. reals is characterized by computable Lipschitz functions, we use lower and upper semi-computable Lipschitz functions to characterize Solovay reducibility for computably approximable reals. Roughly speaking, α is Solovay reducible to β if and only if (β, α) is sandwiched between two such functions.

1.3. **Characterization via signed-digit representation.** The next question discussed in this paper is whether the Solovay reducibility for computably approximable reals is characterized by bounded use.

Recall that the original Solovay reducibility is well-behaved only within left-c.e. reals. Does a reducibility notion exist such that it is well-behaved for all reals and has similar properties to those of Solovay reducibility and preferably coincides with Solovay reducibility for left-c.e. reals? Moreover, it would be desirable to be able to define such a reducibility by bounded use like tt-reducibility and wtt-reducibility.

One such trial would be computable Lipschitz reducibility (abbreviated by cL-reducibility), which is defined by a Turing reduction with bounded use $n + O(1)$. The n-th bit is computable from $(n + c)$-bits of the oracle. Although cL-reducibility shares some traits with Solovay reducibility, the two are not comparable [3, Theorem 9.1.6, 9.10.1].

The issue is not with reducibility but with representation. When representing a real number using binary representation, even if the real is found to be included in a short interval with rational endpoints, this may not determine a long initial segment of the binary representation of the real.

The same issue arises in the field of computable analysis [17], which studies the computability of analysis. The signed-digit representation is one of the most common representations of real numbers used in computable analysis, which best fits the paper's purpose. While the binary representation uses $\{0, 1\}$ to represent each digit, the signed-digit representation uses $\{0, 1, -1\}$. This allows for overlapping the set of cylinders represented by finite digits and is suitable for defining the use of oracles.

We answer the question above. Solovay reducibility can be characterized via Turing reductions with bounded use with respect to the signed-digit representation (Theorem 4.10). Thus, if one replaces binary representation in cL-reducibility with signed-digit representation, it characterizes Solovay reducibility.

This result is pleasing to us. This characterization does not rely on approximations of reals; hence, we offer a natural extension of Solovay reducibility for all reals. Future work will explore its properties.

1.4. **Separation.** In the characterization of Solovay reducibility for computably approximable reals by Lipschitz functions, we use lower and upper semi-computability. In the characterization for left-c.e. reals, we used intervals whose upper endpoints are left-c.e. Are these specific notions truly indispensable? We aim to address this query. To be more specific, will the notion change if one substitutes computable functions for lower and upper semi-computable functions in the characterization? Will the notion change if one requires the functions in the characterization to be locally defined?

To tackle these questions, we present simpler variants of Solovay reducibility, specifically cL-local reducibility and cL-open reducibility. We then separate between these variants in Theorem 5.4 and 5.5, which means that the simpler notions do not characterize Solovay reducibility.

1.5. **Overview.** An overview of this paper is as follows. Starting with Section 3, we delve into characterizing Solovay reducibility for computably approximable reals, achieved through Lipschitz functions. To give it a proof, we also provide a Cauchy-style characterization. In Section 4, we characterize Solovay reducibility via Turing reductions with bounded use with respect to the signed-digit representation. In Section 5, we introduce some variants of Solovay reducibility and separate them.

2. Preliminaries

We follow the standard notation from computability theory, computable analysis, and algorithmic randomness. For details, see such as Soare [15], Brattka, Hertling, and Weihrauch [1], and Downey and Hirschfeldt [3], respectively.

2.1. **Computability of reals.** A function $f :\subseteq 2^{<\omega} \to 2^{<\omega}$ is a *partial computable function* if it is computable by a Turing machine. A real x is *computable* if there exists a computable sequence $(a_n)_n$ of rationals such that $|a_{n+1} - a_n| < 2^{-n}$ for all $n \in \omega$ and $x = \lim_n a_n$. A real x is *left-c.e.* if there exists an increasing computable sequence $(a_n)_n$ of rationals such that $x = \lim_n a_n$. A real x is *right-c.e.* if $-x$ is left-c.e. A real x is *weakly computable* if there exists a computable sequence $(a_n)_n$ of rationals such that $\sum_n |a_{n+1} - a_n| < \infty$ and $x = \lim_n a_n$, or equivalently, there are two left-c.e. reals y, z such that $x = y - z$. A real x is *computably approximable* if there exists a computable sequence $(a_n)_n$ of rationals such that $x = \lim_n a_n$. The set of all computable reals, all left-c.e. reals, all weakly computable reals, and all computable approximable reals are denoted by **EC**, **LC**, **WC**, and **CA**, respectively. We have the following inclusions:

$$\mathbf{EC} \subsetneq \mathbf{LC} \subsetneq \mathbf{WC} \subsetneq \mathbf{CA},$$

and each inclusion is proper.

An open set on $\mathbb{R}$ is called *c.e.* if it is the union of a computable sequence of intervals (a, b) with rational endpoints. More precisely, an open set $U \subseteq \mathbb{R}$ is c.e. if there exist computable sequences $(a_n)_n, (b_n)_n$ of rationals such that $a_n < b_n$ for all $n \in \omega$ and $U = \bigcup_n (a_n, b_n)$. The sequence $((a_n, b_n))_n$ of pairs is called a representation of U. A function $f : \mathbb{R} \to \mathbb{R}$ is *lower semi-computable* if $f(x)$ can be computably approximated from below from a representation of x. This is equivalent to saying that $\{x \in \mathbb{R} \ : \ f(x) > q\}$ is a c.e. open set uniformly in $q \in \mathbb{Q}$, in the sense that their representations are computable uniformly in q. A function $f : \mathbb{R} \to \mathbb{R}$ is *upper semi-computable* if $-f$ is lower semi-computable. A function $f : \mathbb{R} \to \mathbb{R}$ is called *computable* if it is lower and upper semi-computable.

2.2. **Solovay reducibility.** The original definition of Solovay reducibility is defined as follows. Let α, β be left-c.e. reals. Then $\alpha \le_S \beta$ if there are a constant $q \in \omega$ and a partial computable function $f : \mathbb{Q} \to \mathbb{Q}$ such that if $r \in \mathbb{Q}$ and $r < \beta$, then $f(r) \downarrow < \alpha$ and $\alpha - f(r) < q(\beta - r)$ where $\downarrow$ means "is defined" (Solovay [16]).

Let $(a_n)_n, (b_n)_n$ be computable increasing sequences of rationals converging to α, β, respectively. Then, $\alpha \le_S \beta$ if and only if there are a constant q and a computable function g such that $\alpha - a_{g(n)} < q(\beta - b_n)$ for all n (Calude, Coles, Hertling, and Khoussainov [2]). In particular, $\alpha \le_S \beta$ if and only if there are computable increasing sequences $(a_n)_n, (b_n)_n$ of rationals converging to α, β respectively and a constant q such that $\alpha - a_n < q(\beta - b_n)$ for all n.

Solovay completeness has a strong connection to Martin-Löf randomness (ML-randomness) as follows. We say that a left-c.e. real β is Solovay complete for left-c.e. reals if $\alpha \le_S \beta$ for all left-c.e. reals α. Then, the Kučera-Slaman theorem [8] states that a left-c.e. real is *Solovay complete* if and only if it is ML-random. We usually denote a left-c.e. ML-random real by Ω.

Solovay reducibility for weakly computable reals is defined as follows.

Definition 2.1 (Zheng and Rettinger [18, Definition 3.1]). Let $\alpha, \beta \in \mathbf{CA}$. Then $\alpha \leq_S \beta$ if there are computable sequences $(a_n)_n, (b_n)_n$ of rationals converging to α, β respectively and a constant q such that $|\alpha - a_n| < q(|\beta - b_n| + 2^{-n})$ for all n.

Intuitively, $\alpha \leq_S \beta$ means that if one is given a good approximation b_n of β, then one can compute a good approximation a_n of α. The approximation $(b_n)_n$ of β may oscillate, so b_n may happen to be very close to β. In such a case, the error of a_n is only as small as 2^{-n} up to a constant. Notice that $(a_n)_n, (b_n)_n$ need not to be increasing. If α, β are left-c.e., then Definition 2.1 coincides with the original definition by Solovay (Zheng and Rettinger [18, Theorem 3.2]). This notion offers better behavior for real numbers that are not left-c.e.

Solovay completeness for weakly computable reals remains the same. In fact, any weakly computable ML-random real is left-c.e. or right-c.e. (Rettinger and Zheng [13, Theorem 2.5]). Furthermore, we have the following:

Proposition 2.2 (Rettinger and Zheng [13, Theorem 3.7, Corollary 3.8]). *For any left-c.e. ML-random real Ω, we have*

$$\{\alpha \in \mathbf{CA} \ : \ \alpha \leq_S \Omega\} = \mathbf{WC}.$$

Furthermore, a weakly computable real is Solovay complete for weakly computable reals if and only if it is ML-random.

Hence, a weakly computable real is Solovay complete if and only if it is Ω or $-\Omega$.

3. Solovay reducibility via Lipschitz functions

Solovay reducibility for left-c.e. reals has a natural characterization via Lipschitz functions.

Proposition 3.1 (Kumabe et al. [7, Theorem 4.2]). *Let α and β be left-c.e. reals. Then $\alpha \leq_S \beta$ if and only if there exists a computable non-decreasing Lipschitz function f whose domain is $(-\infty, \beta)$ and $\lim_{x \to \beta - 0} f(x) = \alpha$.*

This section characterizes Solovay reducibility for computably approximable reals via Lipschitz functions.

3.1. Cauchy-style characterization of Solovay reducibility.

First, we give a new Cauchy-style characterization of Solovay reducibility for computably approximable reals. The original definition by Zheng and Rettinger uses the difference $|\alpha - a_n|$ between the real α and its approximation a_n.

For left-c.e. reals, a characterization of Solovay reducibility is known that uses the difference $a_{n+1} - a_n$ between two approximations a_n and a_{n+1} as follows: For $\alpha, \beta \in \mathbf{LC}$, $\alpha \leq_S \beta$ if and only if there are computable non-decreasing sequences $(a_n)_n$ and $(b_n)_n$ converging to α and β, respectively, and $q \in \omega$ such that

$$(\forall n) a_{n+1} - a_n < q(b_{n+1} - b_n), \tag{1}$$

from [5]; see also [3, Lemma 9.1.7]. We give a corresponding result for computably approximable reals by means of the difference $a_n - a_k$ between two approximations a_n and a_k, which may not be adjacent. Thus, the characterization is similar to the definition of Cauchy sequences.

Proposition 3.2. *Let $\alpha, \beta \in \mathbf{CA}$. Then, the relation $\alpha \leq_S \beta$ holds if and only if there exist computable sequences $(a_n)_n$ and $(b_n)_n$ converging to α and β respectively and $q \in \omega$ such that*

$$(\forall k, n \in \omega)[k < n \Rightarrow |a_n - a_k| < q \cdot (|b_n - b_k| + 2^{-k})] \tag{2}$$

Remark 3.3. For left-c.e. reals α, β, the statement (1) obviously implies the corresponding stronger statement: $(\forall k, n \in \omega)[k < n \Rightarrow a_n - a_k < q(b_n - b_k)]$. For computably approximable reals α, β, this stronger Cauchy-style statement (2) seems to be required for a characterization of Solovay reducibility. We do not know if (2) can be replaced with an adjacent version $(\forall n)|a_{n+1} - a_n| < q(|b_{n+1} - b_n| + 2^{-n})$ or a similar form.

Many results in this paper seem to be able to be extended to replace Solovay reducibility with quasi-Solovay reducibility [7]. Thus, for later use, we give a proof by showing a lemma with some generalized form.

Assumption 3.4. Assume that $F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a computable function such that

(i) $F(0) = 0$,
(ii) F is increasing.

Note that F is continuous because F is computable.

Notation 3.5. For $\alpha \in \mathbf{CA}$, let $\mathrm{CS}(\alpha)$ denote the set of all computable sequences of rationals converging to α [18].

Lemma 3.6. *Let $\alpha, \beta \in \mathbf{CA}$, $(a_n)_n \in \mathrm{CS}(\alpha)$, and $(b_n)_n \in \mathrm{CS}(\beta)$. Suppose that*

$$(\forall n)|\alpha - a_n| \leq F(|\beta - b_n|) + 2^{-n}, \tag{3}$$

where F satisfies Assumption 3.4. From these sequences, one can construct $(c_n)_n \in \mathrm{CS}(\alpha)$ and $(d_n)_n \in \mathrm{CS}(\beta)$ such that

$$(\forall k, n)[k < n \Rightarrow |c_n - c_k| < F(|d_n - d_k|) + 2^{-k}]. \tag{4}$$

Proof. We begin with construction.

Construction.

We will define $(c_n)_n$ and $(d_n)_n$ as computable subsequences of $(a_n)_n$ and $(b_n)_n$ respectively. More precisely, we construct a computable increasing function $j(n)$ from ω to ω and define $c_n = a_{j(n)}$ and $d_n = b_{j(n)}$ for all n. Let $j(0) := 1, c_0 := a_1$ and $d_0 := b_1$. Assume that we have chosen increasing sequence $j(k)$ for $k < n$ and we have defined $c_k := a_{j(k)}$ and $d_k := b_{j(k)}$. Further, assume that for each $k < n$, we have

$$|\alpha - c_k| < F(|\beta - d_k|) + 2^{-j(k)}. \tag{5}$$

Here, note that $j(k) \geq k + 1$ and thus $2^{-j(k)} < 2^{-k}$. Since F is continuous, all sufficiently large j satisfy the following:

$$|a_j - c_k| < F(|b_j - d_k|) + 2^{-k} \tag{6}$$

for each $k < n$. Hence, we can effectively find a j such that

- j simultaneously satisfy (6) for all $k < n$, and
- $j \geq \max\{j(k) : k < n\} + 1$.

Let $j(n)$ be this j and $c_n := a_{j(n)}$ and $d_n := b_{j(n)}$.

Since $(c_n)_n$ and $(d_n)_n$ are subsequence of $(a_n)_n$ and $(b_n)_n$ respectively, we have $(c_n)_n \in \mathrm{CS}(\alpha)$ and $(d_n)_n \in \mathrm{CS}(\beta)$. Furthermore, the induction hypothesis (5) also holds. Hence, we have the following:

$$(\forall k < n)|c_n - c_k| < F(|d_n - d_k|) + 2^{-k},$$

for all n, which is equivalent to the desired statement. □

Proof of Proposition 3.2.
("if" direction) Suppose that $(a_n)_n \in \mathrm{CS}(\alpha)$, $(b_n)_n \in \mathrm{CS}(\beta)$, and $q \in \omega$ satisfies (2). By letting $n \to \infty$, we have

$$(\forall k)|a_k - \alpha| \leq q \cdot (|b_k - \beta| + 2^{-k}) < (q+1)(|b_k - \beta| + 2^{-k}),$$

unless β is a rational, which implies $\alpha \leq_S \beta$ by Definition 2.1.

("only if" direction) Suppose that $\alpha \leq_S \beta$ via $(a_n)_n$, $(b_n)_n$, and $q \in \omega$. By shifting $(a_n)_n, (b_n)_n$ finitely many times if necessary, say, c times, we can assume to have

$$|\alpha - a_n| < q(|\beta - b_n| + 2^{-n-c}) < q|\beta - b_n| + 2^{-n}$$

for all n. Now we apply Lemma 3.6 with $F(x) = qx$ to deduce the existence of $(c_n)_n \in \mathrm{CS}(\alpha)$ and $(d_n)_n \in \mathrm{CS}(\beta)$ such that, for all $k, n \in \omega$,

$$k < n \Rightarrow |c_n - c_k| < q|d_n - d_k| + 2^{-k}.$$

Hence, the triple of $(c_n)_n$, $(d_n)_n$, and $q+1$ is a witness of (2). □

3.2. **Characterization via Lipschitz functions.** Now, we characterize Solovay reducibility for computably approximable reals via Lipschitz functions.

While Solovay reducibility for left-c.e. reals has a characterization via computable Lipschitz functions in Proposition 3.1, we use upper and lower semi-computable Lipschitz functions for a characterization of Solovay reducibility for computably approximable reals.

Definition 3.7. A *function interval* is the pair of two functions f and h with $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. A function interval (f, h) is *semi-computable* if f is lower semi-computable and h is upper semi-computable.

Theorem 3.8. *Let $\alpha, \beta \in$ **CA**. Then, $\alpha \leq_S \beta$ if and only if there exist a semi-computable function interval (f, h) such that*

(i) *f, h are both Lipschitz functions,*
(ii) *$f(\beta) = h(\beta) = \alpha$.*

In other words, the point (β, α) is sandwiched between f and h, and f and h converges to α as the input variable goes to β. Since f and h are Lipschitz, both functions are continuous. If $f = h$ everywhere, then f and h are computable Lipschitz functions. Thus, the condition used in the theorem above is weaker than the condition being computable Lipschitz functions.

We give a proof of this theorem by giving lemmas. The "if" direction follows from the lemma below by letting $F(x) = qx$ for some $q \in \omega$.

Lemma 3.9. *Let $\alpha, \beta \in$ **CA**. Suppose that there exists a semi-computable function interval (f, h) such that*

(i) $|f(x) - \alpha| \le F(|x - \beta|)$ *for all* $x \in \mathbb{R}$,
(ii) $\lim_{x \to \beta} h(x) = \alpha$.

where F satisfies Assumption 3.4. Then, there exist $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$ such that

$$(\forall n)|\alpha - a_n| < F(|\beta - b_n|) + 2^{-n}.$$

Proof. Suppose such a function interval (f, h) exists. Fix $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$. We further assume that $b_n \neq \beta$ for each n.

We construct a computable increasing function $p : \omega \to \omega$ such that

$$(\forall n)|\alpha - a_{p(n)}| < F(|\beta - b_{p(n)}|) + 2^{-n},$$

which implies the desired claim. Let $p(0) = 0$. Assume that $p(k)$ has been defined for each $k < n$. For each $n \in \omega$, we search $m > p(n-1)$ satisfying

(I) $h(b_m) - f(b_m) < 2^{-n-1}$,
(II) $f(b_m) - 2^{-n-1} < a_m < h(b_m) + 2^{-n-1}$.

Then, let $p(n)$ be this m. Since $a_m \to \alpha$, $b_m \to \beta$ as $m \to \infty$, we have $f(b_m) \to \alpha$ by (i) and the continuity of F by Assumption 3.4. We also have $h(b_m) \to \alpha$ by (ii). Hence, items (I) and (II) hold for all sufficiently large m. Hence, such m is always found.

For this $m = p(n)$, we have

$$f(b_m) - a_m < 2^{-n-1} < 2^{-n}. \tag{7}$$

Here, we used the first inequality of item (II). Similarly, we have the following, where we use (II) for the first inequality, and (I) for the second inequality.

$$a_m - f(b_m) \le h(b_m) + 2^{-n-1} - f(b_m) \le 2^{-n}. \tag{8}$$

Hence, the difference can be evaluated as

$$|\alpha - a_m| \le |\alpha - f(b_m)| + |f(b_m) - a_m| \le F(|\beta - b_m|) + 2^{-n},$$

where we used the triangle inequality for the first inequality and assumption (i) and the conditions (7) and (8) for the second inequality. □

The "only if" direction of Theorem 3.8 follows from the lemma below.

Lemma 3.10. *Let $\alpha, \beta \in$ **CA**, $(a_n)_n \in \mathrm{CS}(\alpha)$, $(b_n)_n \in \mathrm{CS}(\beta)$. Suppose that*

$$(\forall k, n)[k < n \Rightarrow |a_n - a_k| \le F(|b_n - b_k|) + 2^{-k}], \tag{9}$$

where F satisfies Assumption 3.4. Further, assume that F is subadditive:

$$F(x + y) \le F(x) + F(y) \text{ for all } x, y \ge 0.$$

Then, there exists a semi-computable function interval (f, h) such that

(i) $|f(x) - f(y)| \le F(|x - y|)$ *and* $|h(x) - h(y)| \le F(|x - y|)$ *for all* $x, y \in \mathbb{R}$,
(ii) $f(\beta) = h(\beta) = \alpha$.

Proof. The condition (9) is equivalent to

$$(\forall k, n)[k < n \Rightarrow a_k - F(|b_n - b_k|) - 2^{-k} < a_n < a_k + F(|b_n - b_k|) + 2^{-k}].$$

Inspired from this, we define functions f and h as follows:

(a) $f(x) = \sup_{n\in\omega}(a_n - F(|x - b_n|) - 2^{-n})$,
(b) $h(x) = \inf_{n\in\omega}(a_n + F(|x - b_n|) + 2^{-n})$.

Notice that f is lower semi-computable and h is upper semi-computable.

First, we claim that (f, h) is a function interval. Suppose for a contradiction that $h(x) < f(x)$ for some $x \in \mathbb{R}$. Then, there exist $k, n \in \omega$ such that

$$a_k + F(|x - b_k|) + 2^{-k} < a_n - F(|x - b_n|) - 2^{-n}. \tag{10}$$

Then, using the monotonicity and sub-additivity of F, we have

$$F(|b_n - b_k|) \le F(|x - b_n| + |x - b_k|) \le F(|x - b_n|) + F(|x - b_k|),$$
$$F(|b_n - b_k|) + 2^{-\min\{k,n\}} < F(|x - b_n|) + F(|x - b_k|) + 2^{-n} + 2^{-k} < a_n - a_k,$$

which contradicts the assumption (9).

Next, we show item (ii). By definition of f, we have $f(\beta) \ge a_n - F(|\beta - b_n|) - 2^{-n}$ for all n. By letting $n \to \infty$, we have $f(\beta) \ge \alpha$ because F is continuous and $F(0) = 0$. Similarly, we have $h(\beta) \le \alpha$. Then, by $f \le h$ we have $f(\beta) = h(\beta) = \alpha$.

Finally, we show the item (i). We only give a proof for f because the proof for h is analogous. It suffices to show the following:

$$f(y) \le f(x) + F(|x - y|) \text{ for all } x, y \in \mathbb{R},$$

which is implied by

$$a_n - F(|y - b_n|) - 2^{-n} \le a_n - F(|x - b_n|) - 2^{-n} + F(|x - y|) \text{ for all } n,$$

which is equivalent to

$$F(|x - b_n|) \le F(|y - b_n|) + F(|x - y|).$$

If y is between x and b_n, then this is true by the sub-additivity of F. If y is not between x and b_n, then at least one of $|y - b_n|$ and $|x - y|$ is larger than or equal to $|x - b_n|$. Thus, this holds. □

3.3. **Bounded approximation.** We have seen some properties of Solovay reducibility for computably approximable reals via computable approximations in this section. Recall that, by definition, every weakly computable real has a computable approximation with the bounded sum of the differences. Here, we remark that we can enforce the condition in the propositions we have proved in this section.

In addition to Notation 3.5, we use the following notation.

Notation 3.11. For $\alpha \in \mathbf{WC}$, and let $\mathrm{CS}^{\mathrm{bd}}(\alpha) = \{(a_n)_n \in \mathrm{CS}(\alpha) : \sum_n |a_{n+1} - a_n| < \infty\}$, where the superscript "bd" is for "bounded".

First, we enforce this condition for Lemma 3.6, which will imply the other results relatively straightforwardly.

Proposition 3.12. *In Lemma 3.6, we further assume one of the following (i) or (ii);*

(i) $\alpha, \beta \in \mathbf{WC}$,

(ii) $\beta \in \mathbf{WC}$ *and* F *is superadditive:*

$$F(x+y) \geq F(x) + F(y) \text{ for all } x, y \geq 0.$$

Then, we have $\alpha \in \mathbf{WC}$ *and we can enforce* $(c_n)_n \in \mathrm{CS}^{\mathrm{bd}}(\alpha)$ *and* $(d_n)_n \in \mathrm{CS}^{\mathrm{bd}}(\beta)$ *in* (4).

Proof. First, we assume (i). By Lemma 3.6, there are $(c_n)_n \in \mathrm{CS}(\alpha)$ and $(d_n)_n \in \mathrm{CS}(\beta)$ with the inequality (4). Since $\alpha, \beta \in \mathbf{WC}$, there are sequences $(x_n)_n \in \mathrm{CS}^{\mathrm{bd}}(\alpha)$, $(y_n)_n \in \mathrm{CS}^{\mathrm{bd}}(\beta)$.

We construct a strictly increasing computable function $j(n)$ such that

$$(c_{j(n)})_n \in \mathrm{CS}^{\mathrm{bd}}(\alpha),\ (d_{j(n)})_n \in \mathrm{CS}^{\mathrm{bd}}(\beta) \tag{11}$$

To do this, let $j(0) = 1$ and, for each $n \geq 1$, search the index $j > j(n-1)$ such that

$$|c_j - x_j| < 2^{-n},\ |d_j - y_j| < 2^{-n}, \tag{12}$$

and let $j(n)$ be the minimal such index. Since $(d_n)_n$ and $(y_n)_n$ are computable and converge to the same real β, we can always find such j.

Then, the inequality (4) holds for this new pair of sequences. This is because, for each $k, n \in \omega$ such that $k < n$, we have $j(k) < j(n)$,

$$|c_{j(n)} - c_{j(k)}| < F(|d_{j(n)} - d_{j(k)}|) + 2^{-j(k)},$$

and $j(k) > k$.

We claim that (11) holds. This is because

$$\begin{aligned}|d_{j(n+1)} - d_{j(n)}| &\leq |d_{j(n+1)} - y_{j(n+1)}| + |y_{j(n+1)} - y_{j(n)}| + |y_{j(n)} - d_{j(n)}| \\ &< |y_{j(n+1)} - y_{j(n)}| + 2^{-n-1} + 2^{-n},\end{aligned}$$

by the triangle inequality and the inequality (12), whose sum over n is bounded by $(y_n)_n \in \mathrm{CS}^{\mathrm{bd}}(\beta)$. The fact $(c_{j(n)})_n \in \mathrm{CS}^{\mathrm{bd}}(\alpha)$ can be proved similarly.

Now we assume (ii). We similarly construct the function $j(n)$ as above, but we replace the inequality 12 with

$$|d_j - y_j| < 2^{-n}.$$

because we cannot use $(x_n)_n$. We can prove the inequality (4) and the fact $(d_{j(n)})_n \in \mathrm{CS}^{\mathrm{bd}}(\beta)$ similarly.

To prove $(c_{j(n)})_n \in \mathrm{CS}^{\mathrm{bd}}(\alpha)$, let $M \in \omega$ be such that

$$\sum_n |d_{j(n+1)} - d_{j(n)}| < M.$$

Then, we have

$$|c_{j(n+1)} - c_{j(n)}| < F(|d_{j(n+1)} - d_{j(n)}|) + 2^{-j(n)},$$

whose sum from $n = 0$ to $N \in \omega$ is

$$\begin{aligned}\sum_{n=0}^{N} |c_{j(n+1)} - c_{j(n)}| &< \sum_{n=0}^{N} F(|d_{j(n+1)} - d_{j(n)}|) + \sum_{n=0}^{N} 2^{-j(n)} \\ &\leq F(\sum_{n=0}^{N} |d_{j(n+1)} - d_{j(n)}|) + \sum_{n=0}^{N} 2^{-n-1} \\ &< F(M) + 1\end{aligned}$$

by the superadditivity and the monotonicity of F. Since N is arbitrary, we have $(c_{j(n)})_n \in \mathrm{CS}^{\mathrm{bd}}(\alpha)$. $\square$

Proposition 3.12 implies the followings:

Corollary 3.13. *For $\alpha \in \mathbf{CA}$, $\beta \in \mathbf{WC}$ such that $\alpha \leq_S \beta$, we have $\alpha \in \mathbf{WC}$.*

Corollary 3.14. *In Proposition 3.2 and Definition 2.1, if we further assume that $\beta \in \mathbf{WC}$, we can enforce $(a_n)_n \in \mathrm{CS}^{\mathrm{bd}}(\alpha)$ and $(b_n)_n \in \mathrm{CS}^{\mathrm{bd}}(\beta)$.*

4. Solovay reducibility via signed-digit representation

In this section, we give another characterization of Solovay reducibility via Turing reduction with bounded use with respect to signed-digit representation. The condition is similar to the condition of cL-reducibility, except that the cL-reducibility uses a binary representation while our characterization of Solovay reducibility uses signed-digit representation.

4.1. Computable-Lipschitz reducibility on Cantor space.

The definition of cL-reducibility is as follows.

Definition 4.1 (Downey, Hirschfeldt, and LaForte [4]). For $\alpha, \beta \in 2^\omega$, the real α is computably-Lipschitz-reducible to β, denoted by $\alpha \leq_{cL}^{2^\omega} \beta$, if there exists a Turing functional Φ such that $\alpha = \Phi(\beta)$ and $\mathrm{use}(\Phi, \beta, n) \leq n + O(1)$ where use is the use function.

See [3, Section 9.6] for the terminology of this notion.

The cL-reducibility is similar to Solovay reducibility in some sense. For example, both Solovay and cL-reducibility imply K-reducibility. However, cL-reducibility and Solovay reducibility are incomparable even for left-c.e. reals.

Theorem 4.2 ([3, Theorem 9.1.6]). *There exist left-c.e. reals $\alpha \leq_S \beta$ such that $\alpha \not\leq_{wtt} \beta$.*

Note that cL-reducibility implies wtt-reducibility.

Theorem 4.3 ([3, Theorem 9.10.1]). *There exist left-c.e. reals $\alpha \leq_{cL}^{2^\omega} \beta$ such that $\alpha \not\leq_S \beta$.*

Notice that cL-reducibility uses the binary expansion of the reals α, β as representation.

4.2. **Signed-digit representation.** Here, we review some notions from computable analysis and introduce the signed-digit representation.

Let Σ be an alphabet. Let Σ^* denote the set of all finite sequences of Σ and Σ^ω the set of all infinite sequences of Σ. A representation ρ of $\mathbb{R}$ is a surjective function $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$.

The most well-studied representation would be the Cauchy representation, which is an infinite sequence of rationals converging to the real fast enough. Here, we use the signed-digit representation ρ_{sd} [17, Section 7.2]: Let $\Sigma = \{0, \pm 1\}$ and

$$\rho_{sd} : \Sigma^\omega \to [-1, 1], \quad \rho_{sd}(X) = \sum_{n=0}^{\infty} X(n) 2^{-n-1}.$$

One can extend this by adding integer parts so that the range is the whole real line $\mathbb{R}$. However, we use this ρ_{sd} for simplicity.

The cylinder sets induced by this representation are also defined as follows: For a finite sequence $\sigma \in \Sigma^*$, let $[\sigma]_{sd}$ be the set of reals ρ_{sd}-represented the sequences with initial segment σ, that is,

$$[\sigma]_{sd} = \{\rho_{sd}(X) \in [0, 1] \ : \ \sigma \prec X \text{ for some } X \in \Sigma^\omega\}.$$

For example, $[0]_{sd} = [-1/2, 1/2]$, $[(-1)]_{sd} = [-1, 0]$, and $[10]_{sd} = [1/4, 3/4]$.

We use the notation $X \upharpoonright n$ for $X \in \Sigma^\omega$ and $n \in \omega$ to mean the initial segment of X with length n, that is,

$$X \upharpoonright n = X(0)X(1) \cdots X(n-1).$$

One desirable property of the signed-digit representation is the following:

Observation 4.4. *Let $n \in \omega$. For any interval $I \subseteq [-1, 1]$ with length $|I| \leq 2^{-n}$, there exists $\sigma \in \Sigma^*$ with length $|\sigma| = n$ such that*

$$I \subseteq [\sigma]_{sd}.$$

For any string $\sigma \in \Sigma^*$ with length $|\sigma| = n$, $[\sigma]_{sd}$ is the closed interval with length 2^{-n+1} whose endpoints have the form $k \cdot 2^{-n}$ for some integer k. Furthermore, two adjacent cylinder sets made from strings of length n overlap by the length of 2^{-n}. Thus, we can find such a string.

If the endpoints of I are rational, one can find such I effectively. Further, we can do this uniformly in the following sense.

Proposition 4.5. *Let $(I_n)_n$ be a computable sequence of intervals of rational endpoints such that*

$$I_n \supseteq I_{n+1}, \ |I_n| \leq 2^{-n}$$

for all n. Then, there exists a computable sequence $X \in \Sigma^\omega$ such that

$$I_n \subseteq [X \upharpoonright n]_{sd}$$

for all n.

As an example, consider the case that

$$I_1 = [1/4, 3/4], \ I_2 = [3/8, 5/8].$$

By Observation 4.4, we can find a string $\sigma_1 = 1$ such that $I_1 \subseteq [\sigma_1]_{sd}$. Since $I_2 \subseteq I_1 \subseteq [\sigma_1]_{sd}$, one can find $\sigma_2 = 10$ such that $\sigma_1 \prec \sigma_2$ and $I_2 \subseteq [\sigma_2]_{sd}$.

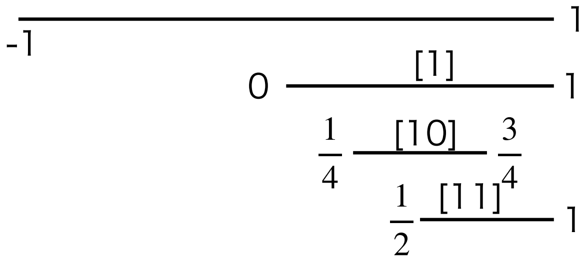

FIGURE 1. The cylinders deduced from sd-representation

We use this fact to show that, for a given ρ_{sd}-representation X_1 of $x \in \mathbb{R}$, we can construct another ρ_{sd}-representation X_2 of $x \in \mathbb{R}$ such that, if another real y is close to x, then some ρ_{sd}-representation of y shares many initial segments with X_2. To formalize this, we define the following notation.

For a string $\sigma \in \Sigma^*$, let $a, b \in \mathbb{R}$ be such that $[\sigma]_{sd} = [a, b]$ and

$$I = [a - 2^{-|\sigma|},\ b + 2^{-|\sigma|}] \cap [-1, 1] \tag{13}$$

Then, I is the closed interval, and the length of this interval is bounded from above as follows:

$$|I| \leq (b - a) + 2^{-|\sigma|+1} = 2^{-|\sigma|+2}.$$

By Observation 4.4, there exists $\tau \in \Sigma^*$ with length $|\sigma| - 2$ such that

$$I \subseteq [\tau]_{sd}, \tag{14}$$

if $|\sigma| \geq 2$. We call this τ a *covering string* of σ and denote it by σ^c. The covering string may not be unique; we may impose further conditions in a proof later.

Proposition 4.6. *From a ρ_{sd}-representation $X \in \Sigma^\omega$ of a real $x \in \mathbb{R}$, one can compute another ρ_{sd}-representation $X' \in \Sigma^\omega$ of the same real $x \in \mathbb{R}$ such that*

$$[x - 2^{-n-3}, x + 2^{-n-3}] \cap [-1, 1] \subseteq [X' \upharpoonright n]_{sd}.$$

Furthermore, $X' \upharpoonright n$ depends only on $X \upharpoonright (n + 3)$.

We call the sequence X' above a covering sequence of x.

Proof. Let

$$I_n = [(X \upharpoonright (n + 3))^c]_{sd}.$$

We can make I_n be decreasing because so is $([X \upharpoonright (n + 3)]_{sd})_n$. Since $|I_n| \leq 2^{-n}$, by relativized Proposition 4.5, one can compute $X' \in \Sigma^\omega$ from X such that $I_n \subseteq [X' \upharpoonright n]_{sd}$ for all $n \in \omega$. Let $a, b \in \mathbb{R}$ such that $x \in [X \upharpoonright (n + 3)]_{sd} = [a, b]$. Then, we have

$$[x - 2^{-n-3}, x + 2^{-n-3}] \cap [-1, 1] \subseteq [a - 2^{-n-3}, b + 2^{-n-3}] \cap [-1, 1] \subseteq I_n$$

because of the property (13) and (14) of the covering string $(X \upharpoonright (n + 3))^c$. $\square$

4.3. **Characterization via bounded use.** A function $g :\subseteq [-1,1] \to [-1,1]$ is called (ρ_{sd}, ρ_{sd})-computable if there exists a Turing reduction $\Phi :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that

$$\rho_{sd}(\Phi(X)) = g(\rho_{sd}(X)) \text{ for all } X \in \mathrm{dom}(\rho_{sd}).$$

In this case, we also say that Φ (ρ_{sd}, ρ_{sd})-realizes g. In other words, given any ρ_{sd}-representation X of a point $x = \rho_{sd}(X) \in \mathrm{dom}(g)$, Φ computes some ρ_{sd}-representation $\Phi(X)$ of $g(x)$.

Definition 4.7. Let $H : \omega \to \omega$ be a non-decreasing function. A function $g :\subseteq \mathbb{R} \to \mathbb{R}$ is (ρ_{sd}, ρ_{sd})-computable with use bound H if there exists a Turing machine $\Phi :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that

(i) Φ (ρ_{sd}, ρ_{sd})-realizes g,
(ii) $\mathrm{use}(\Phi, X, n) \leq H(n)$ for all $X \in \Sigma^\omega$ and $n \in \omega$.

Remark 4.8. The $\mathrm{use}(\Phi, X, n)$ is defined (possibly ∞) even if $\Phi(X)(n)$ diverges.

With these definitions, we can characterize Solovay reducibility via Turing reductions with bounded use with respect to the signed-digit representation. Before that, we give a total-function version as a warm-up, which will be interesting on its own.

Theorem 4.9. *Let $\alpha, \beta \in [-1,1]$. Then, there exists a total and computable Lipschitz function $g : [-1,1] \to [-1,1]$ such that $g(\beta) = \alpha$ if and only if there exists a total function $g : [-1,1] \to [-1,1]$ such that*

(i) *$g(\beta) = \alpha$,*
(ii) *g is (ρ_{sd}, ρ_{sd})-computable with use bound $H(n) = n + O(1)$.*

The proof idea is as follows. For the "if" direction, consider two close inputs x_1 and x_2. Then, some representations of them share many initial segments. Thus, the output should be close. For the "only if" direction, given a total and computable Lipschitz function, we construct a Turing reduction with bounded use. By the Lipschitz condition, the longer we fix the initial segments of the input, the longer we can fix the initial segments of the output.

Proof. ("if" direction)

It suffices to show that g is a Lipschitz function. Let $x_1, x_2 \in [-1,1]$ be such that $x_1 \neq x_2$. Let $d \in \omega$ be such that

$$2^{-d} < |x_1 - x_2| \leq 2^{-d+1}.$$

Then, there exist $X_1, X_2 \in \Sigma^\omega$ such that

$$\rho(X_1) = x_1,\ \rho(X_2) = x_2, \ \text{ and } X_1 \upharpoonright (d-1) = X_2 \upharpoonright (d-1)$$

by Observation 4.4.

Take $\Phi :\subseteq \Sigma^\omega \to \Sigma^\omega$ that (ρ_{sd}, ρ_{sd})-realizes g and assume that the use is bounded by $H(n) = n + c$. Then, $\Phi(X_1)$ and $\Phi(X_2)$ share the same initial segment of length $d - c - 1$, which means

$$|g(x_1) - g(x_2)| \leq 2^{-d+c+2}.$$

Hence, g is a Lipschitz function.

("only if" direction)

Let g be a total Lipschitz computable function such that $g(\beta) = \alpha$. Take some $c \in \omega$ such that the Lipschitz constant for g is bounded by 2^c.

For $X \in \Sigma^\omega$, we define $\Phi(X)(n)$ inductively on $n \in \omega$. Let

$$S_n = [X \upharpoonright (n + c + 2)]_{sd}, \;\; J_n = [\inf_{x \in S_n} g(x), \;\sup_{x \in S_n} g(x)].$$

As an induction hypothesis, we assume $\Phi(X)(k)$ is defined for each $k < n$. We further assume that

$$J_k \subseteq [\Phi(X) \upharpoonright (k+1)]_{sd}$$

for each $k < n$.

Since the length of the interval S_n is 2^{-n-c-1} and g is Lipschitz with Lipschitz constant 2^c, the length of J_n is bounded by $2^c \cdot 2^{-n-c-1} = 2^{-n-1}$. By the induction hypothesis, the interval $J_n \subseteq J_{n-1}$ is contained in $[\Phi(X) \upharpoonright n]_{sd}$ if $n \geq 1$. Thus, we can define $\Phi \upharpoonright (n+1)$ so that

$$J_n \subseteq [\Phi(X) \upharpoonright (n+1)]_{sd} =: J'_n.$$

We claim that Φ (ρ_{sd}, ρ_{sd})-realizes g. Fix $w \in [-1, 1]$ and fix $W \in \Sigma^\omega$ such that $\rho_{sd}(W) = w$. Since $w \in [W \upharpoonright (n + c + 2)]_{sd}$, we have $g(w) \in J_n \subseteq J'_n$. Since the length of J'_n converges to 0, we have $\rho_{sd}(\Phi(W)) = g(w)$.

Finally, note that the use when computing $\Phi(X) \upharpoonright n$ is $n + c + 1$. □

Now, we give the main claim of this section.

Theorem 4.10. *Let α, β be computably approximable reals. Then, $\alpha \leq_S \beta$ if and only if there exists a partial function $g :\subseteq \mathbb{R} \to \mathbb{R}$ such that*

(i) $g(\beta) = \alpha$,
(ii) *g is (ρ_{sd}, ρ_{sd})-computable with use bound $H(n) = n + O(1)$.*

The proof idea of the "if" direction is as follows. While the proof of Theorem 4.9 constructed a computable Lipschitz function, the following proof of Theorem 4.10 constructed computable approximations of α and β. Given some Turing reduction Φ and some approximation b_n of β, we construct an approximation a_n of α. The function g induced by the Φ may not be defined at other than β. Even if b_n is close to β, for some representation of b_n as input, Φ may produce some initial segments of a representation of a real far from α. However, some representations of b_n share many digits with a representation of β. Thus, for such representation of b_n as input, Φ should produce some initial segments of a representation of β. Thus, we need to take a representation of b_n carefully. This problem is overcome by taking a covering sequence.

As an example, let $\beta = -2^{-n}$ and $b_m = 2^{-n}$ for some large n. Consider the following two ρ_{sd}-representations of b_m:

$$B_m = 1(-1)^{n-1}0^\omega,$$
$$B'_m = 0^{n-1}10^\omega.$$

Also consider the following ρ_{sd}-representation of β:

$$B = 0^{n-1}(-1)0^\omega.$$

Suppose that, for each ρ_{sd}-representation of β, the Turing reduction Φ produces some ρ_{sd}-representation of α. Then, $A = \Phi(B)$ is a ρ_{sd}-representation of α. Since B and B'_m share many initial segments, Φ with input B'_m also shares long initial segments with A. However, we have no information on $\Phi(B_m)$.

Although we do not know where β is, if we use a covering sequence of b_m as input, then the cylinders induced from the output contain all reals that are close to b_m.

Assumption 4.11. Assume that $H : \omega \to \omega$ is an unbounded non-decreasing computable function.

Lemma 4.12. *Let $\alpha, \beta \in \mathbf{CA} \cap [-1, 1]$. Let $H : \omega \to \omega$ satisfy Assumption 4.11. Let F be a function satisfying Assumption 3.4 and*

$$(\forall x > 0)(\forall d \in \omega)\left(x \geq 2^{-H(d+1)-3} \Rightarrow F(x) \geq 2^{-d+1}\right) \tag{15}$$

Suppose that there is a Turing functional $\Phi :\subseteq \Sigma^\omega \to \Sigma^\omega$ such that

(i) *for each ρ_{sd}-representation B of β, the output $\Phi(B)$ is a ρ_{sd}-representation of α,*
(ii) $\mathrm{use}(\Phi, X, n)$ *is bounded by $H(n)$ for all $X \in \Sigma^\omega$ and $n \in \omega$.*

Then, there exist $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$ such that

$$(\forall n)|\alpha - a_n| \leq F(|\beta - b_n|) + 2^{-n+1}.$$

Furthermore, $(a_n)_n$ and $(b_n)_n$ do not depend on F.

Remark 4.13. If we further assume $\alpha, \beta \in \mathbf{WC}$, we can enforce $(a_n)_n \in \mathrm{CS}^{\mathrm{bd}}$ and $(b_n)_n \in \mathrm{CS}^{\mathrm{bd}}$ by Proposition 3.12.

Note that the "if" direction follows from this lemma as follows. Suppose that $H(n) = n + c$. We take $F(x) = 2^{c+5} \cdot x$. Then this F satisfies Assumption 3.4 and:

$$(\forall x > 0)(\forall d \in \omega)\left(x \geq 2^{-(d+1+c)-3} \Rightarrow F(x) = 2^{c+5}x \geq 2^{-d+1}\right).$$

By the lemma above, we have $\alpha \leq_S \beta$.

Proof. Let $(b_m)_m \in \mathrm{CS}(\beta)$. We also assume that $0 < |\beta - b_m| < 2^{-H(0)-3}$ for all m.

For each $m \in \omega$, by Proposition 4.6, one can compute $B_m \in \Sigma^\omega$ such that

$$\rho_{sd}(B_m) = b_m,\ [b_m - 2^{-n-3}, b_m + 2^{-n-3}] \cap [-1, 1] \subseteq [B_m \restriction n]_{sd}. \tag{16}$$

We define a computable sequence $(a_n)_n$ of Σ^* with an increasing computable function $p(n)$ inductively on n as follows: Let $p(0) = 0$. For $n \geq 1$, suppose that $p(k)$ is defined for each $k < n$. Search $m \geq \max\{p(n-1)+1, H(n)+3\}$ such that $\Phi(B_m)$ produces at least n-digits. If m is found, then let $p(n)$ be this m and let $A_n = \Phi(B_{p(n)}) \restriction n$ and $a_n = \rho_{sd}(A_n 0^\omega)$.

We claim this procedure works; we can always find such m. Suppose that n is given. Since $b_n \to \beta$ as $n \to \infty$, we have $|\beta - b_m|$ is sufficiently small for large m. For such m, we have $\beta \in [B_m \restriction H(n)]_{sd}$. Thus, B_m and some ρ_{sd}-representation of β share $H(n)$ digits. Hence, $\Phi(B_m \restriction H(n))$ produces at least n-digits by (ii).

We claim that

$$(\forall n)|\alpha - a_n| \leq F(|\beta - b_{p(n)}|) + 2^{-n+1}. \tag{17}$$

Fix $n \in \omega$. Let $d \in \omega$ be such that

$$2^{-H(d+1)-3} \leq |\beta - b_{p(n)}| < 2^{-H(d)-3}. \tag{18}$$

If $d \leq n$, then

$$\beta \in [b_{p(n)} - 2^{-H(d)-3}, b_{p(n)} + 2^{-H(d)-3}] \cap [-1, 1] \subseteq [B_{p(n)} \upharpoonright H(d)]_{sd}$$

by the right inequality of (18) and (16). Since the use of Φ at β is bounded by H, by means of (i) we have $\alpha \in [\Phi(B_{p(n)} \upharpoonright H(d)) \upharpoonright d]_{sd} = [A_n \upharpoonright d]_{sd}$, which implies

$$|\alpha - a_n| \leq 2^{-d+1} \leq F(|\beta - b_{p(n)}|),$$

where the last inequality follows from the left inequality of (18) and the assumption of F, that is, (15). If $d > n$, then $\beta \in [B_{p(n)} \upharpoonright H(n)]_{sd}$ and $|\alpha - a_n| \leq 2^{-n+1}$ by replacing d above with n. Combined with them, we have the inequality (17). □

The proof for the "only if" direction is analogous to that of Theorem 4.9.

Proof of the "only if" direction of Theorem 4.10. Suppose that $\alpha \leq_S \beta$. Then, by Theorem 3.8, there exist a semi-computable function interval (f, h) such that f, h are both Lipschitz functions and $f(\beta) = h(\beta) = \alpha$. Take some $c \in \omega$ such that the Lipschitz constants for f, h are bounded by 2^c.

For $X \in \Sigma^\omega$, we define $\Phi(X) \upharpoonright (n+1)$ inductively on $n \in \omega$. Let

$$S_n = [X \upharpoonright (n + c + 3)]_{sd}, \; J_n = [\min_{x \in S_n} f(x), \max_{x \in S_n} h(x)].$$

Since $f(x) \leq h(x)$ for each $x \in [-1, 1]$, the interval J_n has a positive length or is a single-point set. Since f is continuous and S_n is a compact set, $A = \min_{x \in S_n} f(x)$ exists. Since f is the pointwise limit of increasing sequence $(f_i)_i$ of uniformly computable functions, by Dini's theorem, the convergence is uniform. This implies $A = \sup_i \min_{x \in S_n} f_i(x)$. Hence, A is left-c.e. (One can prove this by assuming only lower semi-continuity of f instead of continuity. See Appendix for the details.) Similarly, $\max_{x \in S_n} h(x)$ exists and is right-c.e. Thus the length $|J_n|$ is a right-c.e. real.

As an induction hypothesis, we assume $\Phi(X) \upharpoonright (k+1)$ is defined for each $k < n$. We further assume that

$$J_k \subseteq [\Phi(X) \upharpoonright (k+1)]_{sd}$$

is confirmed at the stage such that $\Phi(X)(k)$ is defined for each $k < n$. Then, we just wait a stage s such that $|J_n[s]| \leq 2^{-n-1}$ where $J_n[s]$ is the approximation of J_n at stage s. By the induction hypothesis, we have

$$J_n[s] \subseteq J_{n-1}[s] \subseteq [\Phi(X) \upharpoonright n]_{sd}$$

if $n \geq 1$. Thus, we can define $\Phi(X) \upharpoonright (n+1)$ so that

$$J_n \subseteq [\Phi(X) \upharpoonright (n+1)]_{sd} =: J'_n.$$

Let g be a partial function such that $\mathrm{dom}(g) \ni \beta$ and $g(\beta) = \alpha$. We claim that Φ defined above (ρ, ρ)-realizes g. Fix a ρ-representation $B \in \Sigma^\omega$ of β. Since $\beta \in [B \upharpoonright (n + c + 3)]$ and the length of this interval is 2^{-n-c-2}, we have

$$\inf_{x \in S_n} f(x) \geq \alpha - 2^c \times 2^{-n-c-2} = \alpha - 2^{-n-2}.$$

Similarly $\sup_{x \in S_n} h(x) \leq \alpha + 2^{-n-2}$. Thus, $|J_n| \leq 2^{-n-1}$. Hence, $\rho_{sd}(\Phi(B)) \in \mathbb{R}$ is defined. Furthermore, we also have $\alpha \in J_n \subseteq J'_n$ for all n. Hence, $\Phi(X)$ is a ρ-representation of α.

Again, note that the use when computing $\Phi(X) \upharpoonright n$ is $n + c + 2$. □

5. Some variants of Solovay reducibility

Proposition 3.1 characterizes Solovay reducibility for left-c.e. reals via computable Lipschitz functions whose domain is an open interval. Can we extend the function to be total?

In Theorem 3.8, we characterized Solovay reducibility for computably approximable reals via two semi-computable Lipschitz functions. Are these notions different if we require the function to be computable instead of semi-computable?

To answer these questions, we consider some variants of Solovay reducibility and separate the variants and Solovay reducibility.

Definition 5.1 (cL-open reducibility)**.** We say that an open interval $I = (a, b)$ is c.e. if a is a right-c.e. real and b is a left-c.e. real. For $\alpha, \beta \in \mathbb{R}$, α is computably-Lipschitz-reducible to β on a c.e. open interval, denoted by $\alpha \leq^{op}_{cL} \beta$, if there exists a Lipschitz computable function f on a c.e. open interval I such that $\lim_{x \in I \to \beta} f(x) = \alpha$.

Definition 5.2 (cL-local reducibility)**.** For $\alpha, \beta \in \mathbb{R}$, α is computably-Lipschitz-reducible to β locally, denoted by $\alpha \leq^{loc}_{cL} \beta$, if there exists a Lipschitz computable function f on an open interval I such that $\beta \in I$ and $f(\beta) = \alpha$.

For cL-open reducibility, β may be an end-point of I, and it is possible that $\beta \notin I$. For cL-local reducibility, β should be contained in the domain of f.

Notice that we used a total Lipschitz function in Theorem 4.9. Since the function in the definition of cL-local reducibility can be extended to be a total Lipschitz function, the condition used in Theorem 4.9 is equivalent to cL-local reducibility.

Observation 5.3. *For computably approximable reals α, β, we have*

$$\alpha \leq^{loc}_{cL} \beta \Rightarrow \alpha \leq^{op}_{cL} \beta \Rightarrow \alpha \leq_S \beta.$$

For left-c.e. reals α, β, $\alpha \leq^{op}_{cL} \beta$ if and only if $\alpha \leq_S \beta$.

Proof. The first implication follows from the definition.

For the second implication, let f and I be a witness of $\alpha \leq^{op}_{cL} \beta$. Let $(b_n)_n$ be a computable sequence of rationals converging to β. Since β is an accumulation point of I, we can further assume that $b_n \in I$ for all $n \in \omega$. Since f is computable, there exists a computable sequence $(b_n)_n$ of rationals such that

$$|f(a_n) - b_n| < 2^{-n}$$

for all n. Then,

$$|\beta - b_n| = |\lim_n f(a_n) - f(a_n)| + |f(a_n) - b_n| < L|\alpha - a_n| + 2^{-n},$$

where L is a Lipschitz constant of f. Hence, we have $\alpha \leq_S \beta$.

For left-c.e. reals, the "if" direction follows from Proposition 3.1. □

We will prove that these implications are strict below.

5.1. **Separation between cL-loc and cL-open reducibilities.** We prove that cL-loc and cL-open reducibilities differ even for left-c.e. reals.

Theorem 5.4. *There exist left-c.e. reals α, β such that $\alpha \leq_{cL}^{op} \beta$ but $\alpha \not\leq_{cL}^{loc} \beta$.*

The idea of the proof is as follows. We will construct such $\alpha, \beta \in \mathbf{LC}$ in stages.

We use the priority argument with finite injuries. Each requirement R_i states that if the i-th partial computable function f_i is a Lipschitz function with a given Lipschitz constant, $f_i(\beta) \neq \alpha$. These requirements assure $\alpha \not\leq_{cL}^{loc} \beta$.

The strategy for R_i to be satisfied is as follows. If $\alpha \upharpoonright d(i)$ and $\beta \upharpoonright d(i)$ are fixed, then (β, α) is in the larger box $[x_0, x_1] \times [y_0, y_1]$ in Figure 2. Initially, (β, α) is in A in Figure 2.

If f_i is total, then $f_i(x)$ produces approximations within arbitrary precision. When the computation $f_i(x_1)$ produces an approximation within high precision, the requirement R_i requires attention.

The action for R_i to be met is as follows. Let $f = f_i$. If $(x_1, f(x_1))$ is closer to C than to B, then we redefine the initial segments of α and β so that (β, α) is in B in Figure 2. If $(x_1, f(x_1))$ is closer to B than to C, then we set (β, α) is in C in Figure 2.

If f is total, $(x_1, f(x_1))$ is closer to C, and $f(\beta) = \alpha$, then f should have a steep slope, which contradicts that f is a Lipschitz function with a given Lipschitz constant. For the case that $(x_1, f(x_1))$ is closer to B, we can argue similarly.

At each stage, if some requirements require attention, pick the one with the highest priority among those and act for the requirement. The action injures all requirements with lower priority. If R_i is injured, then set (β, α) to be in A, the initial state.

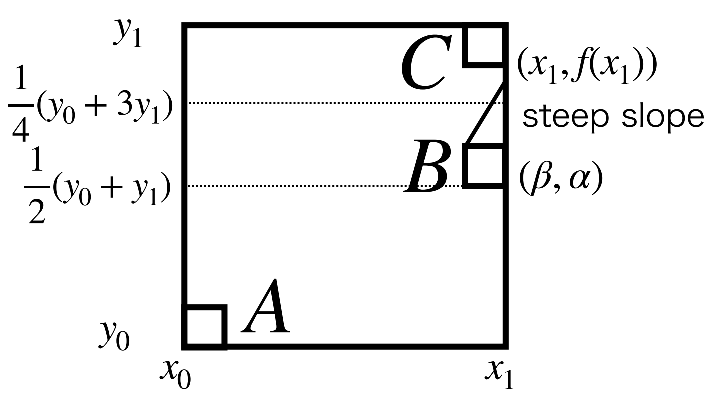

FIGURE 2. possible positions of (β, α)

Proof. We will construct the left-c.e. reals $\alpha, \beta \in [0, 1]$ in stages. The s-th approximations are denoted by infinite binary sequences $\alpha_s, \beta_s \in 2^\omega$ such that $\alpha_s(n) = \beta_s(n) = 0$ for all $n \geq d(s+1)$ where d is defined later. In particular, α_s and β_s only have finite information.

We set each requirement R_i as follows. Fix a computable enumeration $(f_i)_i$ of all partial computable functions from $[0,1]$ to $[0,1]$. As usual, we assume that every function is enumerated infinitely often. Then, set

$$R_i: \ f_i \text{ is defined on } [0,1] \text{ and } f_i \text{ is } 2^i\text{-Lipschitz } \Rightarrow f_i(\beta) \neq \alpha.$$

If $\alpha \leq_{cL}^{loc} \beta$ and $\alpha, \beta \in [0,1]$, then there exists a locally Lipschitz computable function f such that $f(\beta) = \alpha$. Then, there exists an index i such that $f_i = f$ in a neighborhood of β. Hence, meeting all requirements implies $\alpha \not\leq_{cL}^{loc} \beta$. We will enforce $\alpha, \beta \in \mathbf{LC}$ and $\alpha \leq_S \beta$ by the way of construction.

Each requirement R_i manages a predetermined part of α_s and β_s as follows. Let $d : \omega \to \omega$ be a computable function that diverges fast enough with $d(0) = 0$, say, $d(i) = 4i^2$. Let T_i be the finite sequence of consecutive natural numbers from $d(i)$ to $d(i+1) - 1$. We use the following notation: For $X \in 2^\omega$,

$$X \upharpoonright n = X(0)X(1) \cdots X(n-1),$$
$$X \upharpoonright T_i = X(d(i))X(d(i)+1) \cdots X(d(i+1)-1).$$

We refer to $X \upharpoonright T_i$ as T_i-interval of X. The requirement R_i may change T_i-intervals of α_s and β_s.

Note that there is a partial computable function from $(i, d, q) \in \omega \times \omega \times (\mathbb{Q} \cap [0,1])$ to a rational approximation $(f_i \upharpoonright d)(q)$ of $f_i(q)$ within 2^{-d} if defined. Even if $f_i(q)$ is not defined, then $(f_i \upharpoonright d)(q)$ may be defined for some d and may not be defined for other d. Also note that the relation "$(f_i \upharpoonright d)(q)$ is defined within s steps" is decidable.

Construction.

Fix some $i \in \omega$. The strategy for the single requirement R_i to be satisfied is as follows. The requirement R_i only cares about $\alpha_s \upharpoonright T_i$ and $\beta_s \upharpoonright T_i$.

At the initial stage $s = 0$, define

$$\alpha_s \upharpoonright T_i = \beta_s \upharpoonright T_i = 0^{|T_i|} \tag{19}$$

where $|T_i| = d(i+1) - d(i)$. Hence, $\alpha_0 = \beta_0 = 0^\omega$. Notice that the possible position of (β, α) in Figure 2 is in A.

We say that R_i *requires attention* at stage s if both of the following two conditions hold:

(a) R_i is not met at stage $s-1$,

(b) $(f_i \upharpoonright d(i+1))(x_1)$ is defined within s steps,

where x_0, x_1 are the reals whose binary expansions are

$$x_0 = (\beta_{s-1} \upharpoonright d(i))0^\omega, \quad x_1 = (\beta_{s-1} \upharpoonright d(i)))1^\omega.$$

Also note that the conditions (a) and (b) are decidable, respectively. The condition (b) means that $f_i(x)$ is already defined precisely enough to meet R_i by changing T_i-intervals of α_s, β_s.

The *action* for R_i to be met is as follows. Define β_s, α_s as follows:

$$\alpha_s \upharpoonright T_i = 1z^{|T_i|-1}, \quad \beta_s \upharpoonright T_i = 1^{|T_i|}, \tag{20}$$

where

$$z = \begin{cases} 0 & \text{if } (f_i \restriction d(i+1))(x_1) \geq (y_0 + 3y_1)/4, \\ 1 & \text{if } (f_i \restriction d(i+1))(x_1) < (y_0 + 3y_1)/4, \end{cases}$$

$$y_0 = (\alpha_{s-1} \restriction d(i))0^\omega,$$

$$y_1 = (\alpha_{s-1} \restriction d(i))1^\omega.$$

Notice that the possible range of the point (β, α) in Figure 2 moves from region A to region B when $z = 0$ or to region C when $z = 1$.

Note that the side length of the larger square is $2^{-d(i)}$ because we fix the first $d(i)$ bits of α_s and β_s. Similarly, the side length of the smaller square is $2^{-d(i+1)}$.

The priority order is $R_i > R_{i+1}$ for all $i \in \omega$. If some action for a requirement with higher priority than R_i is taken at stage s, then we say that R_i is *injured* at stage s and define T_i-interval as in (19), that is, it goes back to the initial state.

We say that a requirement R_i is *met* at stage s if an action for R_i is taken at or before stage s and R_i is not injured after the action stage until the end of stage s. Notice that each R_i is initially not *met*.

Now, we define the strategies for all R_i simultaneously. At stage $s = 0$, all T_i-intervals of α_s, β_s have the initial state, say, (19). At stage $s \geq 1$, check whether each requirement R_i requires attention for $i \leq s$. If there are no such requirements, then $j(s)$ is not defined and each T_i-intervals are restrained by

$$\alpha_s \restriction T_i = \alpha_{s-1} \restriction T_i, \quad \beta_s \restriction T_i = \beta_{s-1} \restriction T_i. \tag{21}$$

If there are such requirements, then let $j = j(s)$ be the one with the highest priority among those. Then, we act for R_j at this stage s. Thus, R_j is defined by (20). Each requirement R_i for $i > j$ is injured and initialized by (19). Each requirement R_i for $i < j$ restrains the interval by (21).

Verification.

Claim: For each $i \in \omega$, the actions for R_i are taken at most finitely many times.

We can prove this by induction on i. Once the action for R_i is taken, R_i continues to be met until a requirement with higher priority injures R_i. Since the actions for requirements with higher priority are taken at most finitely many times by the induction hypothesis, so are those for R_i.

Claim: The requirement R_i is not met at stage s if and only if $\alpha_s \restriction T_i, \beta_s \restriction T_i$ have the initial state (19).

Fix $i \in \omega$. We prove this by induction on s. At stage $s = 0$, R_i is not met, and the claim follows from the definition (19). Let $s \geq 1$ be the stage. If no action is taken at stage s, then T_i-interval is restrained by (21), and the claim follows from the induction hypothesis. Suppose that the action for R_j is taken at stage s. If $i < j$, then again, T_i-interval is restrained, and the claim follows from the induction hypothesis. If $i = j$, then R_i is met at stage s, and $\alpha_s \restriction T_i, \beta_s \restriction T_i$ do not have the initial state by (20). If $i > j$, then R_i is injured and R_i is not met at this stage s, and $\alpha_s \restriction T_i, \beta_s \restriction T_i$ are initialized by (19), thus the claim follows.

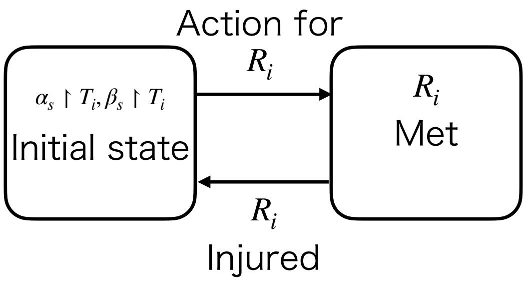

FIGURE 3. possible states of $\alpha_s \upharpoonright T_i, \beta_s \upharpoonright T_i$

Claim: $\alpha, \beta \in \mathbf{LC}$.

Let s be a stage. If no action is taken at stage s, $\alpha_s = \alpha_{s-1}$ and $\beta_s = \beta_{s-1}$. Suppose that the action for R_j is taken at stage s. Then,

$$\alpha_s \upharpoonright T_i = \alpha_{s-1} \upharpoonright T_i, \ \ \beta_s \upharpoonright T_i = \beta_{s-1} \upharpoonright T_i$$

for all $i < j$ by (21). By (20), we have

$$\alpha_s(d(j)) = \beta_s(d(j)) = 1.$$

Since the action for R_j is taken at stage s, R_j is not met at stage $s-1$ by the condition of requiring attention. Hence, T_j-interval of $\alpha_{s-1}, \beta_{s-1}$ have the initial state (19) by the claim above. In particular, we have

$$\alpha_{s-1}(d(j)) = \beta_{s-1}(d(j)) = 0.$$

Hence, α_s, β_s are non-decreasing in s.

Claim: $\alpha \leq_S \beta$.

We evaluate the differences $\alpha - \alpha_s$ and $\beta - \beta_s$.

For each s, let $k(s) = \min\{j(t) \ : \ t > s\}$. This $k(s)$ is the least index of the requirements for actions after stage s. Since $R_{k(s)}$ is not injured after stage s, there is the unique $t > s$ such that $j(t) = k(s)$.

Since any action for R_i where $i < k(s)$ is not taken, the initial $d(k(s))$ bits of approximations of α do not change after stage s. Thus, we have

$$\alpha - \alpha_s \leq 2^{-d(k(s))}. \tag{22}$$

Similarly, the first $d(k(s))$ bits of β_s do not change after stage s. Since the action for $R_{k(s)}$ is taken at stage t, $R_{k(s)}$ is not met at stage $t-1$. If $R_{k(s)}$ is met at stage s, then $R_{k(s)}$ continues to be met at $t-1$ because $R_{k(s)}$ is not injured after stage s, which is a contradiction. Thus, $R_{k(s)}$ is not met at stage s and $\beta_s \upharpoonright T_{k(s)}$ has the initial state at stage s, that is,

$$\beta_s \upharpoonright T_{k(s)} = 0^{|T_{k(s)}|}.$$

Since the action for $R_{k(s)}$ is taken at stage t, and $R_{k(s)}$ is not injured after stage t, $R_{k(s)}$ continues to be met after t, that is,

$$\beta_u \upharpoonright T_{k(s)} = 1^{|T_{k(s)}|}.$$

for all $u \geq t$ by (20) and (21). Thus, we have

$$\beta - \beta_s \geq \sum_{n=d(k(s))}^{d(k(s)+1)-1} 2^{-n-1} - 2^{-d(k(s)+1)} \tag{23}$$

Since $d(i+1) - d(i) \geq 4$ for all $i \in \omega$, we have

$$\beta - \beta_s \geq 2^{-d(k(s))-1} \geq 2^{-1}(\alpha - \alpha_s)$$

by (22) and (23). Hence, the claim follows.

Claim: For each i, the requirement R_i is satisfied.

Fix i such that f_i is defined on $[0,1]$ and $f_i(\beta) = \alpha$. The goal is to show that f_i has a steep slope.

Let $s \in \omega$ be the last stage such that the actions for requirements with higher priority than R_i are taken. If such a stage does not exist, let $s = 0$. Then, R_i is not met at stage s. Notice that R_i continues to be non-met after stage s until the action for R_i is taken. Since f_i is defined on $[0,1]$ as we assumed so above, both conditions (a) and (b) hold eventually. Hence, there is the unique stage $t > s$ such that the action for R_i is taken at stage t.

We claim that the hypothesis $f_i(\beta) = \alpha$ implies that the slope of f_i should be large. Let x_0, x_1, y_0, y_1 be the reals at the stage t. Then, we have

$$x_1 - \beta \leq 2^{-d(i+1)} \tag{24}$$

because they share the same initial segment length $d(i+1)$.

Let $z \in \{0,1\}$ be the one defined at stage t. Suppose that $z = 0$. By (20), the point (β, α) stays in B and

$$\alpha \leq \frac{y_0 + y_1}{2} + 2^{-d(i+1)}.$$

Since $(f_i \upharpoonright d(i+1))(x_1) \geq (y_0 + 3y_1)/4$, we have

$$\begin{aligned} f_i(x_1) - f_i(\beta) &\geq (f_i \upharpoonright d(i+1))(x_1) - 2^{-d(i+1)} - \alpha \qquad (25)\\ &\geq \frac{y_0 + 3y_1}{4} - 2^{-d(i+1)} - \frac{y_0 + y_1}{2} - 2^{-d(i+1)} \\ &= 2^{-d(i)-2} - 2 \cdot 2^{-d(i+1)}. \qquad (26) \end{aligned}$$

By (24) and (26), the slope should be larger than or equal to

$$\frac{2^{-d(i)-2} - 2 \times 2^{-d(i+1)}}{2^{-d(i+1)}} = 2^{d(i+1)-d(i)-2} - 2 = 2^{8i+2} - 2 > 2^i,$$

which contradicts the assumption of R_i. The other case of $z = 1$ is similar.

The final claim above implies $\alpha \not\leq^{loc}_{cL} \beta$. This completes the proof. $\square$

5.2. **Separation between cL-open and Solovay reducibilities.** We have seen that cL-open and Solovay reducibilities are equivalent for left-c.e. reals (Observation 5.3), but they are different for weakly computable reals in general as follows.

Theorem 5.5. *There exist $\alpha, \beta \in$ **WC** such that $\alpha \leq_S \beta$ but $\alpha \not\leq^{op}_{cL} \beta$.*

The proof is similar to that of Theorem 5.4. We focus on the differences and refer to the above for the same argument.

We will construct such $\alpha, \beta \in \mathbf{WC}$ in stages. This time, the approximation can not be monotone, and we use the signed-digit representation ρ_{sd} in the following proof. At each stage s, we define $\alpha_s, \beta_s \in \Sigma^\omega$. For each $n \in \omega$, $\alpha_s(n), \beta_s(n)$ stabilizes as s goes infinity. The induced infinite sequences of Σ are ρ_{sd}-representations of α, β.

We again use the priority argument with finite injuries. Each requirement R_i states that if f_i is defined at all reals less than β or at all reals larger than β, and f_i is a Lipschitz function with a given constant, then $\alpha \leq_{cL}^{op} \beta$ via f_i does not hold.

The strategy for R_i to be satisfied is as follows. If $\alpha_s \upharpoonright d(i)$ and $\beta_s \upharpoonright d(i)$ are fixed, then the point (β, α) is in the larger square in Figure 4. Initially, (β, α) is in E in Figure 4.

We pick rationals x_{-1} and x_1. If the computation $f_i(x_{-1})$ or $f_i(x_1)$ produces an approximation within high precision, then the requirement R_i requires attention.

The action for R_i to be met is as follows. Let $f = f_i$. If the requirement R_i requires attention, we know an approximation with high precision at least one of $f(x_{-1})$ or $f(x_1)$. According to whether the approximation is $\geq y_0$ or $< y_0$, we redefine the initial segments of α_s and β_s so that so that (β, α) is in a smaller square in Figure 4. For example, if the approximation of $f(x_1)$ is larger than y_0, we set (β, α) as the lower right square. If $f(\beta) = \alpha$, then β and x_1 are close while α and $f(x_1)$ are not close, which implies that f should have a steep slope.

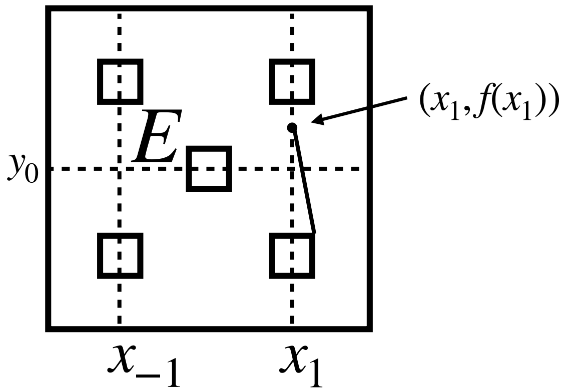

FIGURE 4. possible positions of (β, α)

Proof. We will construct the weakly computable reals $\alpha, \beta \in [-1, 1]$ in stages. The s-th approximations of them are denoted by $\alpha_s, \beta_s \in \Sigma^\omega$ where $\Sigma = \{0, \pm 1\}$.

We set each requirement R_i as follows. Fix a computable enumeration $(f_i)_{i\in\omega}$ of all partial computable functions from $[-1, 1]$ to $[-1, 1]$. As usual, we assume that every function is enumerated infinitely often. Then, set

$$R_i : \ f_i \text{ is defined and } 2^i\text{-Lipschicz on } [-1, \beta) \text{ or } (\beta, 1] \Rightarrow f_i(\beta) \neq \alpha.$$

Then, meeting all requirements implies $\alpha \not\leq_{cL}^{op} \beta$.

Let $d(i)$ and T_i be the same as in the proof of Theorem 5.4. For $X \in \Sigma^\omega$, let

$$X \upharpoonright n = X(0)X(1)\cdots X(n-1), \tag{27}$$
$$X \upharpoonright T_i = X(d(i))X(d(i)+1)\cdots X(d(i+1)-1). \tag{28}$$

Construction.

Fix i. At the initial stage $s = 0$, define

$$\alpha_s \upharpoonright T_i = \beta_s \upharpoonright T_i = 0^{|T_i|}. \tag{29}$$

We say that R_i *requires attention* at stage s if both of the following two conditions hold:

(a) R_i is not met at stage $s-1$,

(b) $(f_i \upharpoonright d(i+1))(x_{-1})$ or $(f_i \upharpoonright d(i+1))(x_1)$ is defined within s steps,

where

$$x_{-1} = \rho_{sd}((\beta_{s-1} \upharpoonright d(i))(-1)0^\omega),$$
$$x_1 = \rho_{sd}((\beta_{s-1} \upharpoonright d(i))10^\omega).$$

The *action* for R_i to be met is as follows. Define

$$\alpha_s \upharpoonright T_i = z0^{|T_i|-1}, \quad \beta_s \upharpoonright T_i = w0^{|T_i|-1}, \tag{30}$$

where

$$y_0 = \rho_{sd}((\alpha_{s-1} \upharpoonright d(i))0^\omega),$$

and z, w are defined below:

	z	w
(c) $(f_i \upharpoonright d(i+1))(x_{-1})$ is defined and $\geq y_0$	-1	-1
(d) $(f_i \upharpoonright d(i+1))(x_{-1})$ is defined and $< y_0$	1	-1
(e) $(f_i \upharpoonright d(i+1))(x_1)$ is defined and $\geq y_0$	-1	1
(f) $(f_i \upharpoonright d(i+1))(x_1)$ is defined and $< y_0$	1	1

If both $(f_i \upharpoonright d(i+1))(x_{-1})$ and $(f_i \upharpoonright d(i+1))(x_{-1})$ are defined, take (e) or (f).

Note that the length of the side of the larger square is $2^{-d(i)+1}$, and the one of the smaller square is $2^{-d(i+1)+1}$.

The priority order is $R_i > R_{i+1}$ for all $i \in \omega$. We use the terminology *injured*, *met*, and *the initial state* similarly in the proof of Theorem 5.4. We also use the same simultaneous strategies for all R_i.

Verification.

Claim: For each i, the actions for R_i are taken at most 2^i many times.

The action for R_0 is taken at most once because it has the highest priority and is not injured. For $i \geq 1$, the number of the actions for R_i is bounded by one plus the number of injuries by requirements with higher priority, which is

$$1 + \sum_{n=0}^{i-1} 2^n = 2^i.$$

Claim: If R_i is not met at stage s, then $\alpha_s \upharpoonright T_i$ and $\beta_s \upharpoonright T_i$ have the initial state (29).

The proof of this part is the same as that in Theorem 5.4.

Claim: $\alpha, \beta \in \mathbf{WC}$.

For each i, we have $j(s) = i$ for at most 2^i many s. For each such s, the difference is bounded by as follows:

$$\begin{aligned} |\rho_{sd}(\alpha_s) - \rho_{sd}(\alpha_{s-1})| &\leq |\rho_{sd}(\alpha_{s-1} \upharpoonright d(i))z0^\omega) - \rho_{sd}(\alpha_{s-1} \upharpoonright d(i))0^\omega)| \\ &\quad + |\rho_{sd}(\alpha_{s-1} \upharpoonright d(i))0^\omega) - \rho_{sd}(\alpha_{s-1})| \\ &\leq 2^{-d(i)+1} + 2^{-d(i+1)+1}. \end{aligned}$$

Recall that if two ρ_{sd}-representations share n digits, then the difference is bounded by 2^{-n+1}. Since the action for R_i is taken at stage s, R_i is not met at stage $s-1$ and $\alpha_{s-1} \upharpoonright T_i$ have the initial state, which implies the last inequality above. Similarly, we have

$$|\rho_{sd}(\beta_s) - \rho_{sd}(\beta_{s-1})| \leq 2^{-d(i)+1} + 2^{-d(i+1)+1}.$$

Thus, the total sums of the differences are bounded by

$$\sum_s |\rho_{sd}(\beta_s) - \rho_{sd}(\beta_{s-1})| \leq \sum_i 2^i \cdot (2^{-d(i)+1} + 2^{-d(i+1)+1}) < \infty.$$

Claim: $\alpha \leq_S \beta$.

We evaluate the differences $|\alpha - \rho_{sd}(\alpha_s)|$ and $|\beta - \rho_{sd}(\beta_s)|$.

For each s, let $k(s) = \min\{j(t) \ : \ t \geq s\}$ and let t be the unique one such that $j(t) = k(s)$.

As in the previous proof, we have

$$\begin{aligned} \alpha_s \upharpoonright d(k(s)) &= \alpha_u \upharpoonright d(k(s)), \\ \beta_s \upharpoonright d(k(s)) &= \beta_u \upharpoonright d(k(s)), \end{aligned}$$

for all $u \geq s$. Thus, we have

$$|\alpha - \rho_{sd}(\alpha_s)| \leq 2^{-d(k(s))+1}. \tag{31}$$

We also have

$$\begin{aligned} \beta_s \upharpoonright T_{k(s)} &= 0^{|T_{k(s)}|}, \\ \beta_u \upharpoonright T_{k(s)} &= w0^{|T_{k(s)}|-1}, \end{aligned}$$

for some $w \in \{\pm 1\}$ for all $u \geq t$.

If $w = 1$, then

$$\begin{aligned} |\rho_{sd}(\beta_s) - \frac{x_{-1} + x_1}{2}| &\leq 2^{-d(k(s)+1)}, \\ |\beta - x_1| &\leq 2^{-d(k(s)+1)}, \\ |x_1 - \frac{x_{-1} + x_1}{2}| &= 2^{-d(k(s))-1}. \end{aligned}$$

Thus, we have

$$|\beta - \rho_{sd}(\beta_s)| \geq 2^{-d(k(s))-1} - 2^{-d(k(s)+1)+1}. \tag{32}$$

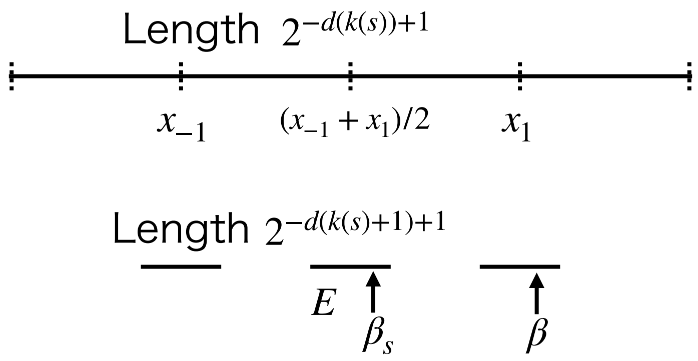

FIGURE 5. possible positions of β_s, β

Since $d(i+1) - d(i) \geq 4$ for all $i \in \omega$, we have

$$|\beta - \rho_{sd}(\beta_s)| \geq 2^{-d(k(s))-2} \geq 2^{-3}|\alpha - \rho_{sd}(\alpha_s)|$$

by (31) and (32). The case $w = -1$ is proved similarly. Hence, the claim follows.

Claim: For each i, the requirement R_i is satisfied.

Fix i such that f_i is defined on $[-1, \beta)$ or $(\beta, 1]$ and $f_i(\beta) = \alpha$. The goal is to show that f_i has a steep slope.

Let s be the last stage such that the action for a requirement with higher priority than R_i is taken. Let $t > s$ be the unique stage such that the action for R_i is taken.

We claim that if $f_i(\beta) = \alpha$, then some slope of f_i should be large. Let x_{-1}, x_1, y_0 be the reals at stage t. Suppose that $z = -1$ and $w = 1$. We have

$$|\beta - x_1| \leq 2^{-d(i+1)+1}. \tag{33}$$

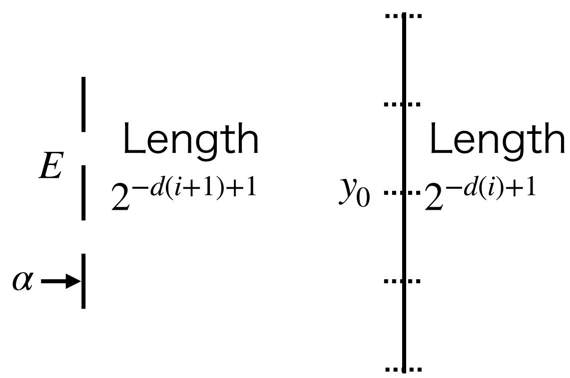

FIGURE 6. possible positions of β_s, β

We also have

$$\alpha \leq y_0 - 2^{-d(i)-1} + 2^{-d(i+1)}.$$

By $f_i(\beta) = \alpha$, we have

$$f_i(x_1) - f_i(\beta) \geq y_0 - 2^{-d(i+1)} - y_0 + 2^{-d(i)-1} - 2^{-d(i+1)} \tag{34}$$
$$= 2^{-d(i)-1} - 2^{-d(i+1)+1}. \tag{35}$$

By (33) and (35), the slope should be larger than

$$\frac{2^{-d(i)-1} - 2^{-d(i+1)+1}}{2^{-d(i+1)+1}} = 2^{d(i)-d(i-1)-2} - 1 > 2^i,$$

which contradicts the assumption of R_i.

The other cases can be proved similarly.

The final claim above implies $\alpha \not\leq_{cL}^{op} \beta$. This completes the proof. □

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Appendix A. Lower semi-computability

The goal of this section is to show the following.

Theorem A.1. *Let $f : [0,1] \to \mathbb{R} \cup \{\infty\}$ be a lower semi-computable function. Then,*

$$A = \min_{x \in [0,1]} f(x)$$

exists and is a left-c.e. real. Here, the value A is possibly infinity.

A.1. **Definition and properties.** Intuitively, a function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is lower semi-computable if $f(x)$ is computably approximable from below from a suitable representation of x. This can be formalized in terms of computable analysis.

A simple formal definition is as follows.

Definition A.2. A function $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is *lower semi-computable* if the sets $\{x \in \mathbb{R} : f(x) > q \in \mathbb{Q}\}$ are c.e. open uniformly in $q \in \mathbb{Q}$.

Another characterization is as follows. A basic open set on $\mathbb{R}$ is an interval with rational endpoints. A rational step function is a function $s : \mathbb{R} \to \mathbb{R}$ such that

$$s(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{B_i},$$

where $a_i \in \mathbb{Q}$, a_i is strictly increasing ($a_1 < a_2 < \cdots < a_n$), and $\bigcup_{i=k}^{n} B_i$ is a finite union of basic open sets for each $k \in \omega$ with $1 \leq k \leq n$. For example, the following function is a rational step function: $n = 2$, $a_1 = 1$, $B_1 = (0,1] \cup [2,3)$, $a_2 = 3$, $B_2 = (1,2)$, which means

$$s(x) = 2 \cdot \mathbf{1}_{(0,1] \cup [2,3)} + 3 \cdot \mathbf{1}_{(1,2)} = \begin{cases} 3 & \text{for } 1 < x < 2 \\ 2 & \text{for } 0 < x \leq 1 \text{ or } 2 \leq x < 3 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that B_1 is not an open set but $B_1 \cup B_2$ and B_2 are finite unions of basic open sets.

Proposition A.3. *A function $f : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is lower semi-computable if and only if there exists a computable sequence $(s_m)_m$ of rational step functions such that*

$$f(x) = \sup_m s_m(x)$$

for every $x \in \mathbb{R}$.

Notice that a rational step function is a lower semi-computable function. Also, note that every lower semi-computable function is lower semi-continuous.

A.2. **Minimality.** The extreme value theorem for lower semi-continuous functions states as follows. We use this theorem to establish the existence part of Theorem A.1.

Theorem A.4. *Let $a, b \in \mathbb{R}$ such that $a < b$ and $f : [a, b] \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous function. Then, f is bounded below and attains its infimum.*

Since every lower semi-computable function is lower semi-continuous, the existence of the minimum in Theorem A.1 follows.

A.3. **Uniform convergence.** A classical Dini's theorem states that, if a monotone sequence of continuous functions converges pointwise on a compact set and if the limit function is also continuous, then the convergence is uniform.

The following theorem is Dini's monotone convergence theorem for semi-continuous functions [14, 17.7.j]. This is sometimes called the Dini-Cartan lemma [6, Lemma 2.2.9].

Theorem A.5. *Let $(f_n)_n$ be a sequence of upper semi-continuous functions from a compact set $X \subseteq \mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$. If $(f_n)_n$ is decreasing and $\lim_n f_n(x) = 0$ pointwise, then the convergence is uniform.*

Corollary A.6. *Let $(f_n)_n$ be a point-wise increasing sequence of lower semi-continuous functions from a compact set $X \subseteq \mathbb{R}$ to $\mathbb{R} \cup \{\infty\}$. Then,*

$$\inf_{x\in X} \sup_n f_n(x) = \sup_n \inf_{x\in X} f_n(x).$$

Proof. For the inequality $\geq$, fix $x \in X$. Then, $\inf_{y\in X} f_n(y) \leq f_n(x)$. Hence, $\sup_n \inf_{y\in X} f_n(y) \leq \sup_n f_n(x)$. Since x is arbitrary, we have $\sup_n \inf_{y\in X} f_n(y) \leq \inf_{x\in X} \sup_m f_n(x)$.

For the converse, let $A = \inf_{x\in X} \sup_n f_n(x)$ and $g_n(x) = A - \min\{A, f_n(x)\}$. Then, $(g_n)_n$ is a sequence of upper semi-continuous functions, decreasing, and $\lim_n g_n(x) = 0$ pointwise. By Dini's theorem for semi-continuous functions, the convergence is uniform.

Let $\epsilon > 0$. Then, there exists $n \in \omega$ such that $g_n(x) < \epsilon$, which implies $A - \epsilon < f_n(x)$ for all $x \in X$. Thus, $A - \epsilon \leq \inf_{x\in X} f_n(x)$. Hence, $A \leq \sup_n \inf_{x\in X} f_n(x)$. ☐

A.4. **Proof.**

Proof of Theorem A.1. Since $A = \min_{x\in[0,1]} f(x)$ exists by Theorem A.4, it suffices to show that $A = \inf_{x\in[0,1]} f(x)$ is left-c.e.

Let $(s_m)_m$ be an increasing computable sequence of rational step functions such that $f(x) = \sup_m s_m(x)$. Since every rational step function is lower semi-continuous, by Corollary A.6, we have $A = \inf_{x\in[0,1]} \sup_m s_m(x) = \sup_m \inf_{x\in[0,1]} s_m(x)$.

Let $g(m) = \inf_{x\in[0,1]} s_m(x)$. Since s_m is a rational step function, $g(m)$ exists for each m, and g is computable. Since $(s_m)_m$ is increasing, so is g. Thus, $A = \sup_m g(m)$ is left-c.e. ☐

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