Strong Solovay reducibility

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Partly joint work with Masahiro Kumabe (Open Univ.) and Toshio Suzuki (TMU).

Abstract

1. We introduce the new notion called “strong Solovay reducibility”.
2. This notion induces many real closed fields.
3. We see some basic properties.
4. We give its characterization via differentiable functions.

Definition

$\mathbf{CA}$: the set of all computably approximable reals or equivalently Δ^0_2

$\mathrm{CS}(\alpha)$: the set of all computable sequence of rationals converging to α

Definition 1

Let $\alpha, \beta \in \mathbf{CA}$. α is strongly Solovay reducible to β, denoted by $\alpha \ll_S \beta$, if there exist $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$ such that

$$\lim_{n\to\infty} \frac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}} = 0.$$

$\alpha \ll_S \beta$ roughly means that the convergence to α is much faster than that to β.

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Real closed field

An ordered field $\mathcal{F}$ is called a real closed field if

- every positive element in $\mathcal{F}$ has a square root in $\mathcal{F}$,
- every odd-degree polynomial with coefficients in $\mathcal{F}$ has a root in $\mathcal{F}$.

Examples

Example 2

1. $\mathbb{R}$
2. the real algebraic numbers (real roots of polynomials of rational coefficients)
3. the computable reals (Rice 1954, Grzegorczyk 1955)
4. the weakly computable reals (Ng 2005, Raichev 2005)
5. the non-random weakly computable reals (Miller 2017)
6. the primitive recursive reals (due to Peter Hertling, see Selivanov and Selivanova 2021)
7. the nearly computable reals (Hertling and Janicki 2023)

Weakly computable reals

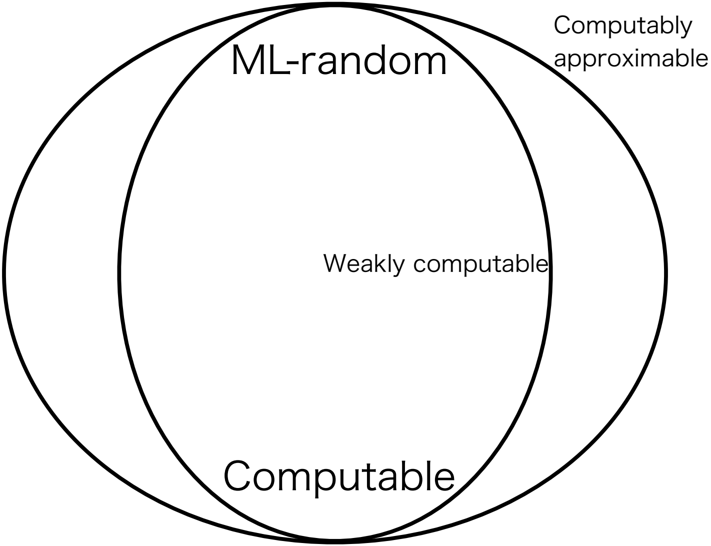
A real $\alpha \in \mathbb{R}$ is computable if there is a computable sequence $(a_n)_n$ of rationals such that $|a_n - a_{n-1}| \leq 2^{-n}$ for all $n \geq 1$ and $\lim_n a_n = \alpha$.

A real $\alpha \in \mathbb{R}$ is left-c.e. if there is a computable sequence $(a_n)_n$ of rationals such that $a_{n-1} \leq a_n$ for all $n \geq 1$ and $\lim_n a_n = \alpha$.

A real $\alpha \in \mathbb{R}$ is weakly computable if there is a computable sequence $(a_n)_n$ of rationals such that $\sum_n |a_n - a_{n-1}| < \infty$ and $\lim_n a_n = \alpha$.

A real is weakly computable if and only if it is the difference of two left-c.e. reals (Ambos-Spies, Weihrauch, and Zheng 2000).

Thus weakly computable reals are sometimes called d.c.e. reals.


Computably
approximable
ML-random
Weakly computable
Computable

Lipschitz continuity

Lemma 3 (Hertling and Janicki 2023 following Raichev 2005)

If a subset $K \subseteq \mathbb{R}$ contains a number $x_0 \neq 0$ and is closed under Lipschitz continuous computable functions $f :\subseteq \mathbb{R}^k \to \mathbb{R}$ with open domain $\mathrm{dom}(f) \subseteq \mathbb{R}^k$, k arbitrary, then K is a real closed subfield of $\mathbb{R}$.

closed under

1. taking a root of a polynomial,
2. Lipschitz continuous computable functions,
3. Solovay reducibility.

Solovay reducibility

Definition 4 (Solovay 1975, Zheng and Rettinger 2004)

Let $\alpha, \beta \in \mathbf{CA}$. $\alpha \leq_S \beta$ if there exist $(a_n)_n \in \mathrm{CS}(\alpha)$, $(b_n)_n \in \mathrm{CS}(\beta)$, and $c \in \omega$ such that

$$|\alpha - a_n| < c(|\beta - b_n| + 2^{-n})$$

for all $n \in \omega$.

$\alpha \leq_S \beta$ roughly means that the convergence to β is not faster than that to α.

Solovay by Lipschitz

Proposition 5 (Kumabe et al. 2020)

Let $\alpha, \beta \in \mathbf{LC}$. $\alpha \leq_S \beta$ if and only if there exists an increasing Lipschitz computable function $f : (-\infty, \beta) \to (-\infty, \alpha)$ such that $\lim_{x \to \beta - 0} f(x) = \alpha$.

Solovay by Lipschitz

A function interval is the pair of two functions f and h with $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. A function interval (f, h) is semi-computable if f is lower semi-computable and h is upper semi-computable.

Theorem 6 (Kumabe, Miyabe, and Suzuki submitted)

Let $\alpha, \beta \in \mathbf{CA}$. $\alpha \leq_S \beta$ if and only if there exists a semi-computable function interval (f, h) such that

1. *f, h are both Lipschitz functions,*
2. *$f(\beta) = h(\beta) = \alpha$.*

RCF by Solovay

Let $S(\beta) = \{\alpha \in \mathbf{CA} \ : \ \alpha \leq_S \beta\}$.

Proposition 7

For every $\beta \in \mathbf{CA}$, $S(\beta)$ forms a real closed field.

$S(\emptyset)$: computable reals

$S(\Omega)$: weakly computable reals

$\{\alpha \in \mathbf{WC} \ : \ \alpha <_S \Omega\}$: non-random weakly computable reals

Not RCF by strict Solovay

Theorem 8 (Downey, Hirschfeldt, and Nies 2002)

Let α be a non-computable non-ML-random left-c.e. real. Then, there are two non-computable left-c.e. reals β and γ such that $\beta, \gamma <_S \alpha$ and $\beta + \gamma = \alpha$.

In particular, for such α, $\{\gamma \in \mathbf{WC} \ : \ \gamma <_S \alpha\}$ does not form a real closed field.

Non-random w.c. reals

Theorem 9 (Demuth 1975)

Let $\alpha, \beta \in \mathbf{LC}$. *Suppose that* $\alpha + \beta$ *is ML-random. Then, at least one of* α *and* β *is ML-random.*

Theorem 10 (Miller 2017)

The set of all non-ML-random weakly computable reals forms a real closed field.

Derivative

Theorem 11 (Barmpalias and Lewis-Pye 2017)

Fix an ML-random left-c.e. real Ω and its approximation $(\Omega_s)_s$. Let α be a weakly computable real with approximation $(\alpha_s)_s$ and let

$$\partial\alpha = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

If α is ML-random, then $\partial\alpha$ exists independent from the approximation and not zero. If α is not ML-random, then $\partial\alpha = 0$.

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Definition

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Definition 12

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$$\lim_{n \to \infty} \frac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}} = 0.$$

$\alpha \ll_S \beta$ roughly means that the convergence to α is much faster than that to β.

RCF by strong Solovay

Proposition 13

$S(\ll \Omega)$ is equal to the set of all non-ML-random weakly computable reals.

Proposition 14

$S(\ll \beta)$ forms a real closed field for every $\beta \in \mathbf{CA}$.

Remark

In Barmpalias and Lewis-Pye's result, the limit does not depend on approximations.

In the definition of strong Solovay reducibility, we only require such an approximation to exist.

Solovay degree invariant

Proposition 15

Let $\alpha, \beta, \gamma \in \mathbf{CA}$.

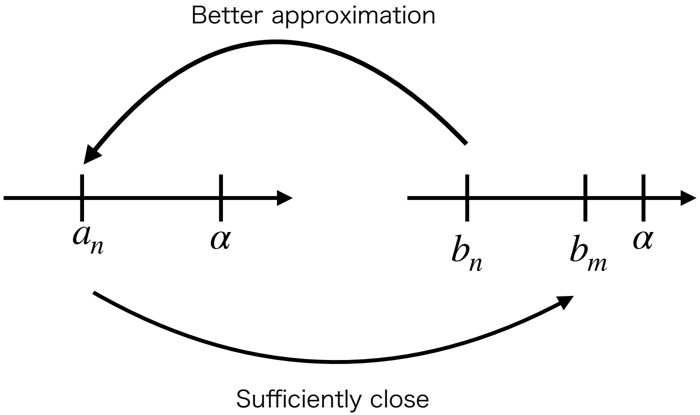
1. If $\alpha \ll_S \beta$, then $\alpha \leq_S \beta$.
2. If $\alpha \leq_S \beta$ and $\beta \ll_S \gamma$, then $\alpha \ll_S \gamma$.
3. If $\alpha \ll_S \beta$ and $\beta \leq_S \gamma$, then $\alpha \ll_S \gamma$.

Thus, $\ll_S$ is Solovay degree invariant.

Reflexivity

Proposition 16

Let $\alpha \in \mathbf{CA}$. Then, $\alpha \ll_S \alpha$ if and only if α is computable.


Better approximation
a_n
α
b_n
b_m
α
Sufficiently close

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Characterization via derivative

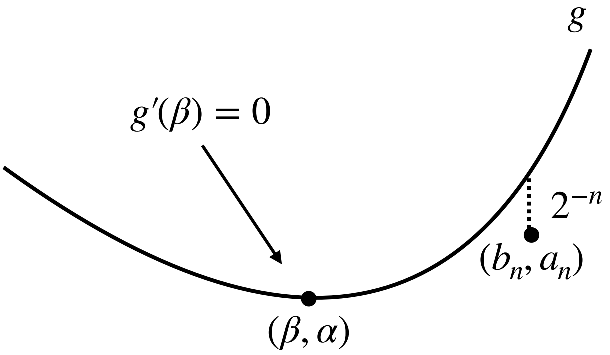
Theorem 17

Let $\alpha, \beta \in \mathbf{CA}$*. Then,* $\alpha \ll_S \beta$ *if and only if there exist* $(a_n)_n \in \mathrm{CS}(\alpha)$*,* $(b_n)_n \in \mathrm{CS}(\beta)$*, and a continuous function* g *such that*

1. *the derivative* $g'(\beta) = 0$*,*
2. $|g(b_n) - a_n| \leq 2^{-n}$ *for all* n*.*

We can further impose the following condition:

- *The function* g *is differentiable on the real line.*


g
$g'(\beta) = 0$
2^{-n}
(b_n, a_n)
(β, α)

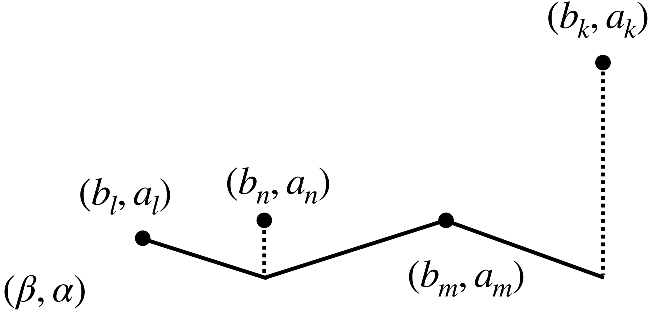
Remark

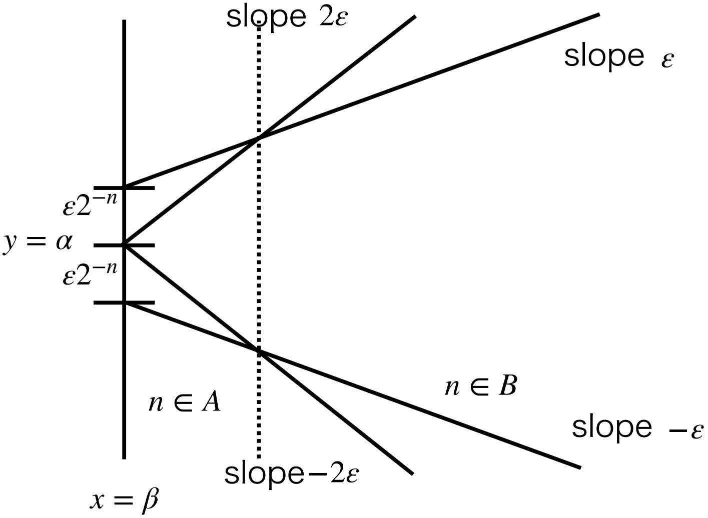
g need not be computable but a_n and $g(b_n)$ should be close.

- strong Solovay: differentiable and zero derivative
- Solovay: Lipschitz continuous
- quasi Solovay: Hölder cotinuous

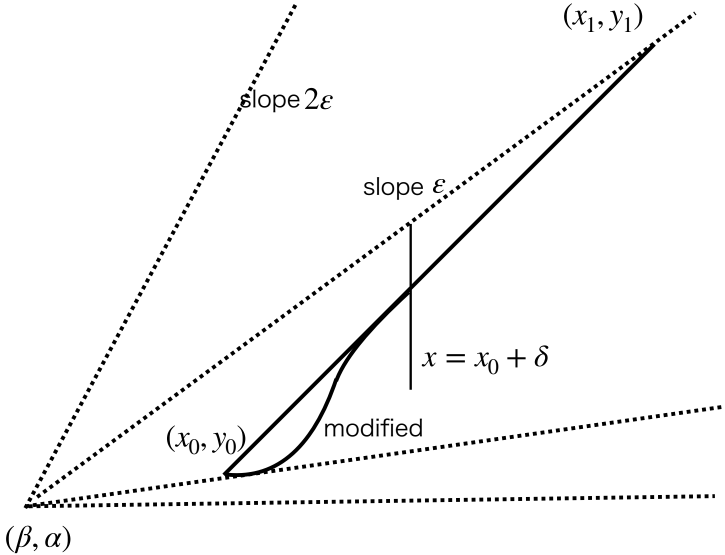
Question 18

Can we impose g to be C^1?


(b_k, a_k)
(b_l, a_l)
(b_n, a_n)
(β, α)
(b_m, a_m)



slope 2ε
slope ε
$\varepsilon 2^{-n}$
$y = \alpha$
$\varepsilon 2^{-n}$
$n \in B$
$n \in A$
slope $-\varepsilon$
slope-2ε
$x = \beta$


(x_1, y_1)
slope 2ε
slope ε
$x = x_0 + \delta$
modified
(x_0, y_0)
(β, α)

Thank you for listening.

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