Strong Solovay reducibility

Kenshi Miyabe (宮部賢志)

Meiji University

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CCR2024@Nagoya

Partly joint work with

- Masahiro Kumabe (Open Univ.)
- Toshio Suzuki (TMU)

Abstract

1. Introduction to algorithmic randomness
2. Closed under “simple” operations
3. Strong Solovay reducibility

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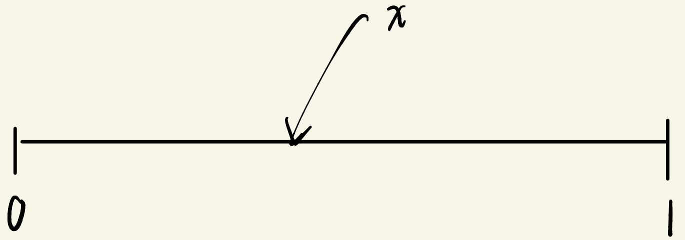
- Real closed field
- Strong Solovay reducibility

Consider sampling from the unit interval.



SLLN

Theorem 1 (Strong Law of Large Numbers, SLLN)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}X(k)=\frac{1}{2}\quad \textit{with probability 1}$$

Here, $X(k)$ is the k-th digit of the infinite binary expansion of $x\in[0,1]$.

The set of sequences that violate the SLLN has measure 0 and is covered by

$$\bigcap_{n\in\omega}U_n$$

where $(U_n)_n$ is a sequence of open sets with measure $\leq 2^{-n}$.

ML-randomness

A c.e. open set is an open set that has the form

$$U = \bigcup_n (a_n, b_n)$$

where $(a_n)_n$ and $(b_n)_n$ be a computable sequence of rationals.

Definition 2

A ML-test is a sequence $(U_n)_n$ of uniformly c.e. open sets such that $\mu(U_n) \leq 2^{-n}$, where μ is the Lebeasgue measure.

A real $x \in [0,1]$ is called ML-random if x is not covered by any ML-test, that is,

$$x \notin \bigcap_n U_n.$$

Path-wise property

Proposition 3

Any ML-random real obeys the strong law of large numbers.

Many limit theorems can be precisely stated in the form:

If a real is sufficiently random, then such-and-such property holds.

Halting probability

A concrete example of ML-random reals:

Example 4 (Halting probability)

$$\Omega = \sum_{\sigma \in \mathrm{dom}(U)} 2^{-|\sigma|}$$

is a left-c.e. ML-random real.

Here, $U :\subseteq 2^{<\omega} \to 2^{<\omega}$ is the universal prefix-free machine.

- It is difficult to capture.
- It can be approximated computably.

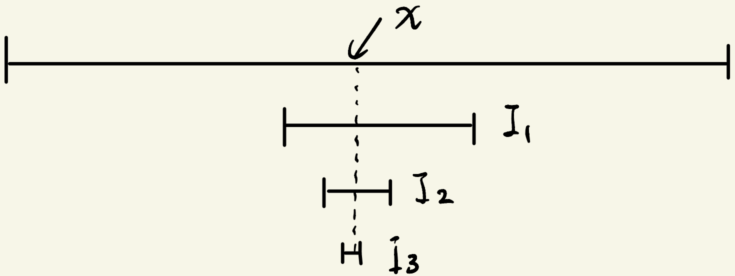
Counterintuitive!!

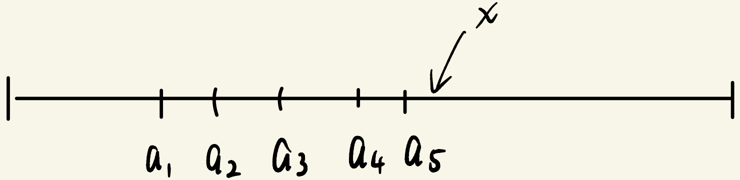
c.a. reals

A real x is called computable if there exists a computable sequence $I_n = (a_n, b_n)$ of intervals with rational endpoints such that $|b_n - a_n| < 2^{-n}$.

A real x is called left-c.e. if there exists an increasing computable sequence a_n of rationals such that

$$x = \lim_n a_n.$$


x
I_1
I_2
I_3


x
a_1
a_2
a_3
a_4
a_5

computably approximable

weakly computable

computable

⋮

3-random

2-random

1-random

Weakly computable reals

A real $\alpha \in \mathbb{R}$ is weakly computable if there is a computable sequence $(a_n)_n$ of rationals such that $\sum_n |a_n - a_{n-1}| < \infty$ and $\lim_n a_n = \alpha$.

A real is weakly computable if and only if it is the difference of two left-c.e. reals (Ambos-Spies, Weihrauch, and Zheng 2000).

Thus weakly computable reals are sometimes called d.c.e. reals.

A real $\alpha \in \mathbb{R}$ is computably approximable if there is a computable sequence $(a_n)_n$ of rationals such that $\lim_n a_n = \alpha$.

Solovay reducibility

Definition 5 (Solovay 1975, Zheng and Rettinger 2004)

Let $\alpha, \beta \in \mathbf{CA}$. $\alpha \leq_S \beta$ if there exist $(a_n)_n \in \mathrm{CS}(\alpha)$, $(b_n)_n \in \mathrm{CS}(\beta)$, and $c \in \omega$ such that

$$|\alpha - a_n| < c(|\beta - b_n| + 2^{-n})$$

for all $n \in \omega$.

$\alpha \leq_S \beta$ roughly means that the convergence to β is not faster than that to α.

This is a reducibility that measures the approximability of reals.

Solovay reducibility and randomness

Proposition 6

Let $\alpha, \beta \in \mathbf{CA}$. $\alpha \leq_S \beta$ implies $\alpha \leq_K \beta$, that is,

$$K(\alpha \upharpoonright n) < K(\beta \upharpoonright n) + O(1)$$

where K is the prefix-free Kolmogorov complexity.

Theorem 7 (Kučera-Slaman theorem)

Let $\alpha \in \mathbf{WC}$. Then, α is ML-random if and only if it is Solovay complete in $\mathbf{WC}$.

Solovay reducibility measures the randomness of reals.

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Closedness

Simple numbers with simple operations produce simple numbers.

- $\mathbb{N}$
- $\mathbb{Z}$
- $\mathbb{Q}$
- the real algebraic numbers

Real closed field

An ordered field $\mathcal{F}$ is called a real closed field if

- every positive element in $\mathcal{F}$ has a square root in $\mathcal{F}$,
- every odd-degree polynomial with coefficients in $\mathcal{F}$ has a root in $\mathcal{F}$.

Real closed field

Example 8

1. $\mathbb{R}$
2. the real algebraic numbers (real roots of polynomials of rational coefficients)
3. the computable reals (Rice 1954, Grzegorczyk 1955)
4. the weakly computable reals (Ng 2005, Raichev 2005)
5. the non-random weakly computable reals (Miller 2017)
6. the primitive recursive reals (due to Peter Hertling, see Selivanov and Selivanova 2021)
7. the nearly computable reals (Hertling and Janicki 2023)

Main question

Are there good subsets of reals such that

- it forms a real closed field,
- it is related to randomness.

Use Solovay reducibility!!

RCF by Solovay

Let $S(\beta) = \{\alpha \in \mathbf{CA} \ : \ \alpha \leq_S \beta\}$.

Proposition 9

For every $\beta \in \mathbf{CA}$, $S(\beta)$ forms a real closed field.

$S(\emptyset)$: computable reals

$S(\Omega)$: weakly computable reals

$\{\alpha \in \mathbf{WC} \ : \ \alpha <_S \Omega\}$: non-random weakly computable reals

Lipschitz continuity

Lemma 10 (Hertling and Janicki 2023 following Raichev 2005)

If a subset $K \subseteq \mathbb{R}$ contains a number $x_0 \neq 0$ and is closed under Lipschitz continuous computable functions $f :\subseteq \mathbb{R}^k \to \mathbb{R}$ with open domain $\mathrm{dom}(f) \subseteq \mathbb{R}^k$, k arbitrary, then K is a real closed subfield of $\mathbb{R}$.

closed under

1. taking a root of a polynomial,
2. Lipschitz continuous computable functions,
3. Solovay reducibility.

Solovay by Lipschitz

Proposition 11 (Kumabe et al. 2020)

Let $\alpha, \beta \in \mathbf{LC}$. $\alpha \leq_S \beta$ if and only if there exists an increasing Lipschitz computable function $f : (-\infty, \beta) \to (-\infty, \alpha)$ such that $\lim_{x \to \beta - 0} f(x) = \alpha$.

Solovay by Lipschitz

A function interval is the pair of two functions f and h with $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. A function interval (f, h) is semi-computable if f is lower semi-computable and h is upper semi-computable.

Theorem 12 (Kumabe, Miyabe, and Suzuki submitted)

Let $\alpha, \beta \in \mathbf{CA}$. $\alpha \leq_S \beta$ if and only if there exists a semi-computable function interval (f, h) such that

1. *f, h are both Lipschitz functions,*
2. *$f(\beta) = h(\beta) = \alpha$.*

Not RCF by strict Solovay

Theorem 13 (Downey, Hirschfeldt, and Nies 2002)

Let α be a non-computable non-ML-random left-c.e. real. Then, there are two non-computable left-c.e. reals β and γ such that $\beta, \gamma <_S \alpha$ and $\beta + \gamma = \alpha$.

In particular, for such α, $\{\gamma \in \mathbf{WC} \ : \ \gamma <_S \alpha\}$ does not form a real closed field.

Non-random w.c. reals

Theorem 14 (Demuth 1975)

Let $\alpha, \beta \in \mathbf{LC}$. Suppose that $\alpha + \beta$ is ML-random. Then, at least one of α and β is ML-random.

Theorem 15 (Miller 2017)

The set of all non-ML-random weakly computable reals forms a real closed field.

Derivative

Theorem 16 (Barmpalias and Lewis-Pye 2017)

Fix an ML-random left-c.e. real Ω and its approximation $(\Omega_s)_s$. Let α be a weakly computable real with approximation $(\alpha_s)_s$ and let

$$\partial\alpha = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

If α is ML-random, then $\partial\alpha$ exists independent from the approximation and not zero. If α is not ML-random, then $\partial\alpha = 0$.

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Definition

$\mathbf{CA}$: the set of all computably approximable reals or equivalently Δ^0_2

$\mathrm{CS}(\alpha)$: the set of all computable sequence of rationals converging to α

Definition 17

Let $\alpha, \beta \in \mathbf{CA}$. α is strongly Solovay reducible to β, denoted by $\alpha \ll_S \beta$, if there exist $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$ such that

$$\lim_{n\to\infty} \frac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}} = 0.$$

$\alpha \ll_S \beta$ roughly means that the convergence to α is much faster than that to β.

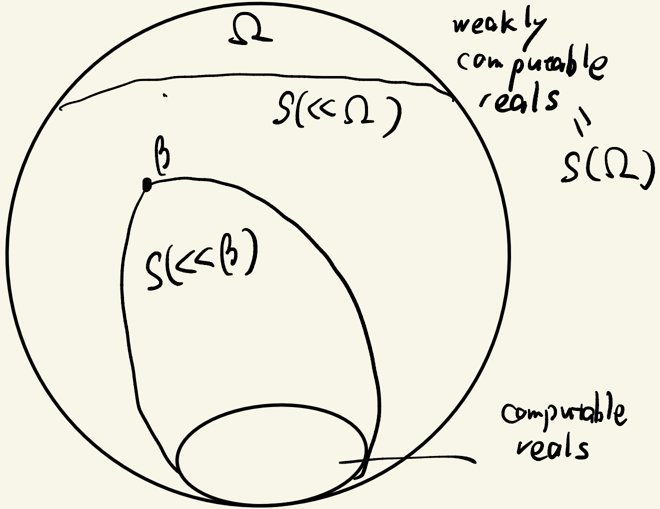
RCF by strong Solovay

Proposition 18

$S(\ll \Omega)$ is equal to the set of all non-ML-random weakly computable reals.

Proposition 19

$S(\ll \beta)$ forms a real closed field for every $\beta \in \mathbf{CA}$.


Ω
weakly computable reals
$=$
$S(\Omega)$
$S(\ll\Omega)$
β
$S(\ll\beta)$
computable reals

Remark

In Barmpalias and Lewis-Pye's result, the limit does not depend on approximations.

In the definition of strong Solovay reducibility, we only require such an approximation to exist.

Solovay degree invariant

Proposition 20

Let $\alpha, \beta, \gamma \in \mathbf{CA}$.

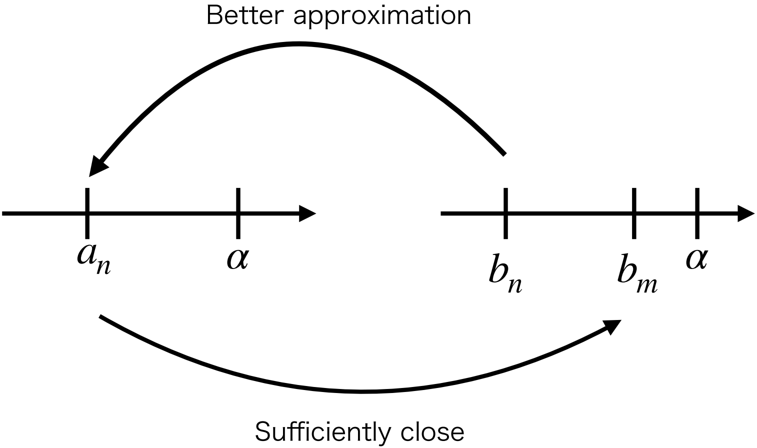
1. If $\alpha \ll_S \beta$, then $\alpha \leq_S \beta$.
2. If $\alpha \leq_S \beta$ and $\beta \ll_S \gamma$, then $\alpha \ll_S \gamma$.
3. If $\alpha \ll_S \beta$ and $\beta \leq_S \gamma$, then $\alpha \ll_S \gamma$.

Thus, $\ll_S$ is Solovay degree invariant.

Reflexivity

Proposition 21

Let $\alpha \in \mathbf{CA}$. Then, $\alpha \ll_S \alpha$ if and only if α is computable.



Better approximation
a_n
α
b_n
b_m
α
Sufficiently close

Characterization via derivative

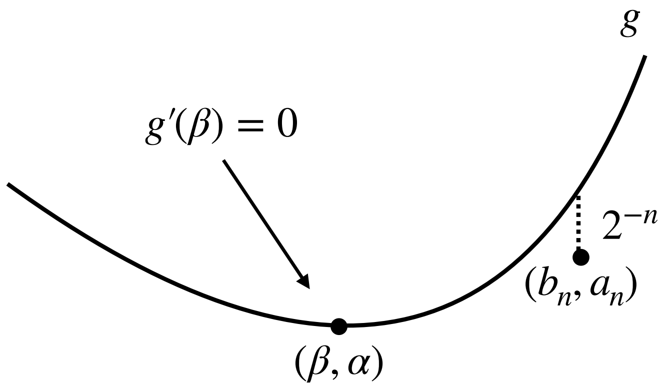
Theorem 22

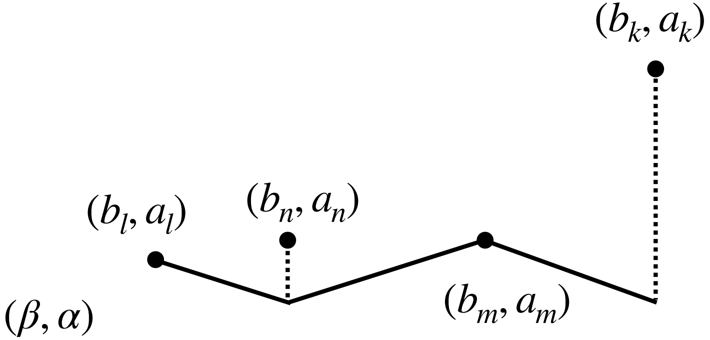
Let $\alpha, \beta \in \mathbf{CA}$*. Then,* $\alpha \ll_S \beta$ *if and only if there exist* $(a_n)_n \in \mathrm{CS}(\alpha)$*,* $(b_n)_n \in \mathrm{CS}(\beta)$*, and a continuous function* g *such that*

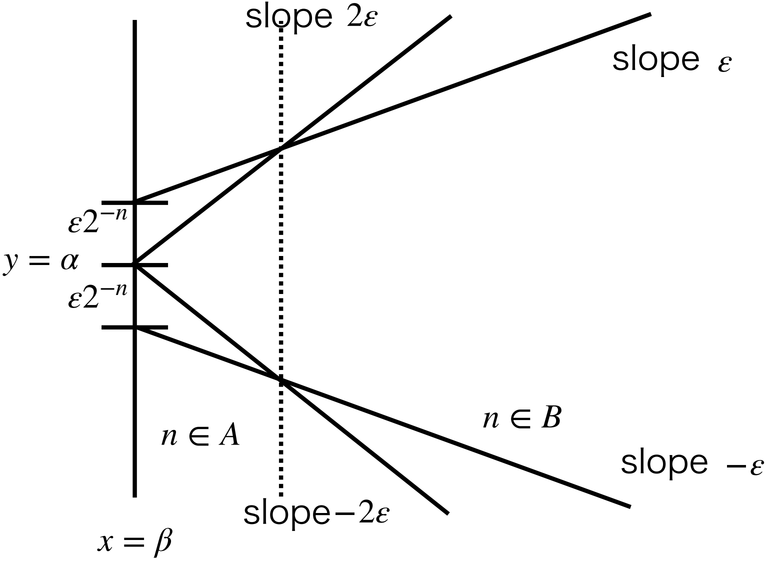
1. *the derivative* $g'(\beta) = 0$*,*
2. $|g(b_n) - a_n| \leq 2^{-n}$ *for all* n*.*

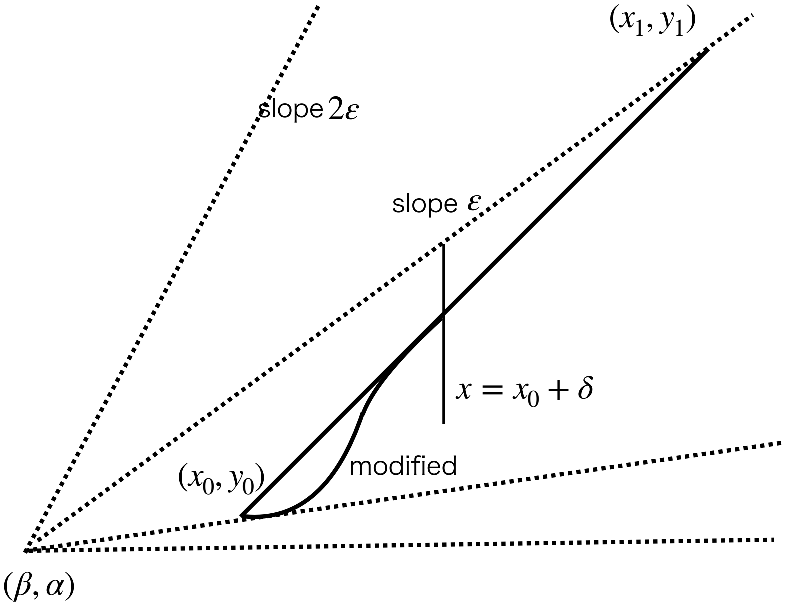
We can further impose the following condition:

- *The function* g *is differentiable on the real line.*


g
$g'(\beta) = 0$
2^{-n}
(b_n, a_n)
(β, α)


(b_k, a_k)
(b_l, a_l)
(b_n, a_n)
(β, α)
(b_m, a_m)


slope 2ε
slope ε
$\varepsilon 2^{-n}$
$y=\alpha$
$\varepsilon 2^{-n}$
$n \in B$
$n \in A$
slope $-\varepsilon$
slope-2ε
$x=\beta$


(x_1, y_1)
slope 2ε
slope ε
$x = x_0 + \delta$
modified
(x_0, y_0)
(β, α)

Thank you for listening.

明治大学
MEIJI UNIVERSITY