# REAL CLOSED FIELDS VIA STRONG SOLOVAY REDUCIBILITY

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ABSTRACT. In computability theory, we naturally encounter some countable real closed fields. Notably, the set of all computable reals forms a real closed field. The main goal of this paper is to explore more deeply the relationship between nonrandomness and real closed fields, by providing various classes of nonrandom reals that form real closed fields via Solovay reducibility.

Solovay reducibility is a popular tool for studying the complexity of certain real numbers in algorithmic randomness theory. In our earlier studies, we explored the relationship between Solovay reducibility and Lipschitz functions, providing an analytical expression and facilitating connections with other concepts in analysis.

We present the hierarchy of Solovay reducibility variants that correspond to different levels of smoothness in real functions such as differentiable functions, Lipschitz functions, and Hölder continuous functions. Using this type of reducibility, we successfully construct numerous countable real closed fields, thus establishing a significant connection between Solovay reducibility and classical algebraic structures.

We also investigate the Solovay reducibility of computably approximable reals without remainder terms.

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## 1. Introduction

1.1. **Real closed field in computability theory.** Some real numbers have a succinct representation, while others do not. Typically, real numbers with a succinct representation maintain their succinctness under basic mathematical operations. To illustrate, rational numbers remain rational when operated upon using quadratic functions with rational coefficients. Algebraic real numbers are the real solutions to algebraic equations with rational coefficients. Real solutions to equations with algebraic real numbers as coefficients are again algebraic real numbers. This property is often expressed by saying that the set of algebraic real numbers forms a *real closed field*.

In computability theory, we naturally encounter some countable real closed fields. Notably, the set of all *computable reals*, which is recognized as a fundamental concept in computability theory, forms a real closed field (Rice [19] and Grzegorczyk [7]). The set of all *weakly computable reals*, which represents a broader class of real numbers than that of computable reals and is also known as d.c.e. reals, forms a real closed field, too (Ng [15] and Raichev [17]). Using the limit lemma, we obtain a further example of a real closed field, which is the set of all *computably approximable reals*: These reals are also known as limit computable reals or $\Delta^0_2$, and the set of these reals forms an even wider class of reals. The set of $K$-trivial reals is an interesting subclass of weakly computable reals, which forms a real closed field; see [16, Corollary 5.5.15] and a comment below it. Recent studies have shown that both primitive recursive reals [20] and nearly computable reals [8] form real closed fields.

1.2. **Solovay reducibility and variants.** The algorithmic randomness theory provides precise definitions of random reals, which are the opposite of real numbers with a scccinct representation. In fact, non-ML-random weakly computable reals form a real closed field, as shown by Miller [13]. In other words, taking real solutions of algebraic equations with non-ML-random weakly computable reals only produces non-ML-random weakly computable reals.

The main goal of this paper is to explore more deeply the relationship between non-randomness and real closed fields, by providing further classes of non-ML-random reals that form real closed fields via Solovay reducibility.

Solovay reducibility is a tool for comparing two real numbers regarding their approximability or randomness. By fixing a particular real number, we demonstrate that the set of real numbers below it in terms of Solovay degrees forms a real closed field (Theorem 3.1). This indicates that Solovay reducibility exhibits desirable properties. As we will see in Subsection 3.2 if we change "below" to "strictly below," the set ceases to be a real closed field.

In our previous study, we examined the relationship between Solovay reducibility and Lipschitz functions. What happens to the corresponding reduction when we vary the smoothness of a function? Recall that if a real function on a closed interval is $C^1$, it is Lipschitz continuous, which implies Hölder continuity. We try to provide a framework within which strong Solovay reducibility (see below), Solovay reducibility, and quasi-Solovay reducibility (see below) can be understood in a unified style.

We define strong Solovay reducibility in Definition 3.8, a novel concept introduced in this paper. We see its basic properties in Subsection 3.3, and we show that strong Solovay reducibility produces real closed fields in Theorem 3.13.

While quasi Solovay reducibility had been defined for left-c.e. reals in our previous work [10, 9], in Section 4, we extend its definition to computably approximable reals in Definition 4.6. We characterize it via Hölder continuous functions in Proposition 4.8. We show that quasi Solovay reducibility produces real closed fields in Proposition 4.10.

In Section 5, we investigate Solovay reducibility of computably approximable reals in a form without remainder terms. We characterize strong Solovay reducibility via derivative in Theorem 5.2.

In Section 6, we investigate the relation between (strong) Solovay reducibility and (strong) $K$ reducibility.

## 2. Preliminaries

We follow the standard notation from computability theory, computable analysis, and algorithmic randomness. For details, see such at Soare [21], Brattka, Hertling, and Weihrauch [3], and Downey and Hirschfeldt [5], respectively.

### 2.1. Computability of reals.

A real $x$ is *computable* if there exists a computable sequence $(a_n)_n$ of rationals such that $|a_{n+1} - a_n| < 2^{-n}$ for all $n \in \omega$ and $x = \lim_n a_n$. A real $x$ is *left-c.e.* if there exists an increasing computable sequence $(a_n)_n$ of rationals such that $x = \lim_n a_n$. A real $x$ is *right-c.e.* if $-x$ is left-c.e. A real $x$ is *weakly computable* if there exists a computable sequence $(a_n)_n$ of rationals such that $\sum_n |a_n - a_{n-1}| < \infty$ and $x = \lim_n a_n$. A real is weakly computable if and only if it is the difference between two left-c.e. reals, thus it is also called a d.c.e. real. A real $x$ is *computably approximable* if there exists a computable sequence $(a_n)_n$ of rationals such that $x = \lim_n a_n$. The set of all computable reals, all weakly computable reals, and all computably approximable reals are denoted by **EC**, **WC**, and **CA**, respectively. We have the following inclusions:

$$\mathbf{EC} \subsetneq \mathbf{WC} \subsetneq \mathbf{CA},$$

and each inclusion is proper.

### 2.2. Solovay reducibility.

For a brief history of Solovay reducibility for computable approximable reals, see our previous paper [11, Section 2].

The following definition is due to [24, Definition 3.1].

**Definition 2.1.** Let $\alpha, \beta$ be computably approximable reals. Then $\alpha \leq_S \beta$ if there are computable sequences $(a_n)_n, (b_n)_n$ of rationals converging to $\alpha, \beta$ respectively and a constant $c \in \omega$ such that $|\alpha - a_n| < c(|\beta - b_n| + 2^{-n})$ for all $n$.

The characterization of Solovay completeness for weakly computable reals via Martin-Löf randomness (ML-randomness) was developed in stages. Solovay [22] showed that each Solovay complete left-c.e. real is ML-random. Kučera and Slaman [12] showed that each left-c.e. ML-random real is Solovay complete. Rettinger and Zheng [18, Corollary 3.8] extended the previous results to weakly computable reals as follows.

**Proposition 2.2.** *A weakly computable real is Solovay complete for weakly computable reals if and only if it is a left-c.e. or right-c.e. ML-random real.*

This theorem shows that, for weakly computable reals, Solovay reducibility captures both the degree of randomness and the degree of approximability.

## 3. Real closed fields of computably approximable reals

An ordered field $F$ is called *real closed* if

(i) any non-negative element $x \geq 0$ in $F$ has a square root in $F$, and
(ii) any odd-degree polynomial with coefficients in $F$ has a root in $F$.

In this section, we show that some classes of reals form real closed fields.

3.1. **Solovay belowness.** For each $\beta \in \mathbf{CA}$, let $S(\leq \beta)$ be the set of all computably approximable reals Solovay reducible to $\beta$, that is,

$$S(\leq \beta) = \{\alpha \in \mathbf{CA} \ : \ \alpha \leq_S \beta\}.$$

If $\alpha \in \mathbf{CA}$, $\beta \in \mathbf{WC}$, and $\alpha \leq_S \beta$, then $\alpha \in \mathbf{WC}$ (see [11, Corollary 3.12]). Hence, if $\beta \in \mathbf{WC}$, then $S(\leq \beta) \subseteq \mathbf{WC}$ and

$$S(\leq \beta) = \{\alpha \in \mathbf{WC} \ : \ \alpha \leq_S \beta\}.$$

If $\beta$ is a computble real, then $S(\leq \beta) = \mathbf{EC}$. Since any ML-random left-c.e. real $\Omega$ is Solovay complete for weakly computable reals (Proposition 2.2), we have $S(\leq \Omega) = \mathbf{WC}$. Recall that $\mathbf{EC}$ and $\mathbf{WC}$ form real closed fields, respectively, as stated in Section 1. We will show that this is true for any $\beta \in \mathbf{CA}$.

**Theorem 3.1.** *Let $\beta$ be a computably approximable real. Then, $S(\leq \beta)$ forms a real closed field.*

Rettinger and Zheng [18, Corollary 3.6] have already shown that $S(\leq \beta)$ is a field for any $\beta \in \mathbf{CA}$. Thus, this is a slight extension of their result.

The key notion for the proof is local Lipschitz continuity. A function $f :\subseteq \mathbb{R}^n \to \mathbb{R}$ is called *locally Lipschitz* if, for any $x \in \mathrm{dom}(f)$, there is a neighborhood $U$ of $x$ and a Lipschitz constant $L > 0$ such that $(\forall \mathbf{u}, \mathbf{v} \in U)[|f(\mathbf{u}) - f(\mathbf{v})| \leq L \cdot ||\mathbf{u} - \mathbf{v}||]$ where $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, and $||\cdot||$ is a norm on $\mathbb{R}^n$. Since all norms on $\mathbb{R}^n$ are equivalent, one can use any norm, say, $||\mathbf{u} - \mathbf{v}|| = \sum_{i=1}^n |u_i - v_i|$.

**Lemma 3.2** (Hertling and Janicki [8] following Raichev [17])**.** *If a subset $K \subseteq \mathbb{R}$ contains a number $x_0 \neq 0$ and is closed under Lipschitz continuous computable functions $f :\subseteq \mathbb{R}^k \to \mathbb{R}$ with open domain* $\mathrm{dom}(f) \subseteq \mathbb{R}^k$*, $k$ arbitrary, then $K$ is a real closed subfield of $\mathbb{R}$.*

The set $S(\leq \beta)$ satisfies the condition.

**Lemma 3.3** (Rettinger and Zheng [18, Theorem 3.5])**.** *For a computably approximable real $\beta$, the set $S(\leq \beta)$ is closed under each locally Lipschitz computable function.*

Since every Lipschitz continuous function with open domain is locally Lipschitz continuous, we have shown Theorem 3.1.

3.2. **Strict Solovay belowness.** Recall the following result stated in Section 1.

**Theorem 3.4** (Miller [13])**.** *The set of all non-ML-random weakly computable reals forms a real closed field.*

Let $\Omega$ be a left-c.e. ML-random real. By Proposition 2.2, the set $S(< \Omega) = \{\alpha \in \mathbf{WC} : \alpha <_S \Omega\}$ is the set of all non-ML-random weakly computable reals. Hence, Miller's result can be rephrased as that $S(< \Omega)$ forms a real closed field.

The first question here is whether we can replace this $\Omega$ by any computably approximable real as we did in Theorem 3.1. The answer is negative, which follows from the following results. See also Downey and Hirschfeldt [5, Theorem 9.5.9, Theorem 9.5.3].

**Theorem 3.5** (Demuth [4])**.** *If $\alpha$ and $\beta$ are two left-c.e. reals such that $\alpha + \beta$ is ML-random, then at least one of $\alpha$ and $\beta$ is ML-random.*

**Theorem 3.6** (Downey, Hirschfeldt, and Nies [6])**.** *If $\alpha$ is a non-computable non-ML-random left-c.e. real, then there are two non-computable left-c.e. reals $\beta$ and $\gamma$ such that $\beta, \gamma <_S \alpha$ and $\beta + \gamma = \alpha$.*

Thus, for any non-computable non-ML-random left-c.e. real $\beta$, the set $S(< \beta) = \{\alpha \in \mathbf{WC} : \alpha <_S \beta\}$ does not form a field, nor let alone form a real closed field. We do not know a condition that $S(< \beta)$ forms a real closed field for other reals $\beta$.

3.3. **Strong Solovay belowness.** The key fact in the proof of Theorem 3.4 is the following.

**Theorem 3.7** (Barmpalias and Lewis-Pye [2]; see Miller [13])**.** *Fix a left-c.e. ML-random real $\Omega$ and its approximation $(\Omega_s)_s$. Let $\alpha$ be a weakly computable real with approximation $(\alpha_s)_s$ and let*

$$\partial\alpha = \lim_{s\to\infty} \frac{\alpha - \alpha_s}{\Omega - \Omega_s}.$$

*If $\alpha$ is ML-random, then $\partial\alpha$ exists independent from the approximation and not zero. If $\alpha$ is not ML-random, then $\partial\alpha = 0$.*

Inspired by this result, we introduce strong Solovay reducibility $\ll_S$ and show that the set $S(\ll \beta)$ (defined below) forms a real closed field for any computably approximable real $\beta$. The terminology of strong Solovay reducibility comes from strong $K$-reducibility $\ll_K$. We will discuss the relationship between them later.

**Definition 3.8.** Let $\alpha, \beta \in \mathbf{CA}$. The real $\alpha$ is *strongly Solovay reducible* to $\beta$, denoted by $\alpha \ll_S \beta$, if there exist computable sequences $(a_n)_n$ and $(b_n)_n$ of rationals converging to $\alpha$ and $\beta$ respectively such that

$$\lim_{n\to\infty} \frac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}} = 0.$$

This condition comes from Definition 2.1 and Theorem 3.7.

**Remark 3.9.** In [9], we introduced a similar but different notion and used the same terminology. In defining strong Solovay reducibility, several variations can be considered. For instance, in the aforementioned definition, we can consider versions where the existential quantifiers for sequences $(a_n)_n$ and $(b_n)_n$ are replaced either by universal quantifiers for both or for one of them. If only one is changed, the order can also be chosen. In this paper, we adopted the definition that allows for the subsequent characterization, but we have not extensively explored other possible variations.

We begin with basic observations.

**Proposition 3.10.** *Let $\alpha, \beta$, and $\gamma$ be computably approximable reals.*

(i) *If $\alpha \ll_S \beta$, then $\alpha \leq_S \beta$.*
(ii) *If $\alpha \leq_S \beta$, $\beta \ll_S \gamma$ then $\alpha \ll_S \gamma$.*
(iii) *If $\alpha \ll_S \beta$ and $\beta \leq_S \gamma$, then $\alpha \ll_S \gamma$.*

In particular, if $\alpha \equiv_S \beta$ and $\beta \ll_S \gamma$, then $\alpha \ll_S \gamma$. If $\alpha \ll_S \beta$ and $\beta \equiv_S \gamma$, then $\alpha \ll_S \gamma$. Thus, the relation $\ll_S$ is Solovay degree invariant.

*Proof.* Straightforward. □

**Proposition 3.11.** *Let $\alpha$ be a computably approximable real. Then, $\alpha \ll_S \alpha$ if and only if $\alpha$ is computable.*

The proof idea is as follows. Suppose that $\alpha \ll_S \alpha$ via $(a_n)_n$ and $(b_n)_n$. If a good approximation $b_n$ is given, then $a_n$ for the same index $n$ is a better approximation. By searching $n'$ such that $a_n$ and $b_{n'}$ are close, we will get a better approximation $b_{n'}$. We can repeat this process and get a better approximation of $\alpha$ as close as one wants.

*Proof.* The "if" direction is obvious.

For the "only if" direction, suppose that $\alpha \ll_S \alpha$ for a real $\alpha \in \mathbf{CA}$. Then, there are computable sequences $(a_n)_n$ and $(b_n)_n$ of rationals both converging to $\alpha$ such that

$$|\alpha - a_n| < c(|\alpha - b_n| + 2^{-n}), \quad \text{where } c = \frac{1}{4}. \tag{1}$$

From these sequences, we construct a computable increasing sequence $(n(k))_k$ such that $|\alpha - b_{n(k)}| < 2^{-k}$ for all $k$, which implies that $\alpha$ is computable.

**Construction.**

Let $n(0)$ be such that $n(0) > 2$ and $|\alpha - b_{n(0)}| < 1$.

Given $n(k-1)$, let $n(k)$ be the smallest $n$ such that

$$n > n(k-1), \; |b_n - a_{n(k-1)}| < 2^{-k-1}. \tag{2}$$

**Verification.**

We show that

(i) one can compute $n(k)$ from $n(k-1)$ if $k \geq 1$,
(ii) $n(k) > k+2$,
(iii) $|\alpha - b_{n(k)}| < 2^{-k}$,

by induction on $k$.

The claim for $k=0$ is true because $b_n \to \alpha$ as $n \to \infty$.

Suppose that $n(k-1)$ is given and $n(k-1) > k+1$ and $|\alpha - b_{n(k-1)}| < 2^{-k-1}$ by induction hypothesis. By inequality (1), we have

$$|\alpha - a_{n(k-1)}| < c(2^{-k-1} + 2^{-n(k-1)}) < c2^{-k} = 2^{-k-2}.$$

Thus, for any sufficiently large $n$, we have

$$|b_n - a_{n(k-1)}| \leq |b_n - \alpha| + |\alpha - a_{n(k-1)}| < 2^{-k-1}.$$

Hence, we can find $n$ satisfying (2) computably, which ensures condition (i) and (ii). Furthermore, we have

$$|\alpha - b_{n(k)}| \leq |\alpha - a_{n(k-1)}| + |a_{n(k-1)} - b_{n(k)}| < 2^{-k-2} + 2^{-k-1} < 2^{-k},$$

which ensures condition (iii). □

For $\beta \in \mathbf{CA}$, let

$$S(\ll \beta) = \{\alpha \in \mathbf{CA} \ : \ \alpha \ll_S \beta\}.$$

**Proposition 3.12.** *The set $S(\ll \Omega)$ is equal to the set of all non-ML-random weakly computable reals.*

*Proof.* Suppose that $\alpha \in S(\ll \Omega)$. Then, $\alpha \ll_S \Omega$ by definition. Since $\ll_S$ implies $\leq_S$ by Proposition 3.10, we have $\alpha \leq_S \Omega$. Since $\Omega \in \mathbf{WC}$, we have $\alpha \in \mathbf{WC}$.

If $\alpha$ is ML-random, we have $\alpha \not\ll_S \alpha$ by Proposition 3.11. Since $\ll_S$ is Solovay degree invariant, we have $\alpha \not\ll_S \Omega$, which is a contradiction. Hence, $\alpha$ is not ML-random.

Suppose that $\alpha$ is a weakly computable real that is not ML-random. Let $(a_n)_n$ be a computable sequence of rationals converging to $\alpha$ such that $\sum_n |a_{n+1} - a_n| < \infty$. Let $(\Omega_n)_n$ be a computable increasing sequence of rationals converging to $\Omega$. Then, by Theorem 3.7, we have

$$\lim_n \frac{\alpha - a_n}{\Omega - \Omega_n} = 0.$$

Thus,

$$0 \leq \frac{|\alpha - a_n|}{|\Omega - \Omega_n| + 2^{-n}} \leq \left| \frac{\alpha - a_n}{\Omega - \Omega_n} \right| \to 0$$

as $n \to \infty$. □

Now Theorem 3.4 can be rephrased by that $S(\ll \Omega)$ forms a real closed field. We show that this is true for any $\beta \in \mathbf{CA}$.

**Theorem 3.13.** *The set $S(\ll \beta)$ forms a real closed field for each computably approximable real $\beta$.*

*Proof.* The proof follows a similar approach to that used in Theorem 3.1. It suffices to show that $S(\ll \beta)$ is closed under each locally Lipschitz computable function. We only show this for two-variable functions because the generalization is straightforward.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a locally Lipschitz computable function and $x, y \ll_S \beta$. The goal is to show $f(x, y) \ll_S \beta$. Take computable sequences $(x_n)_n$, $(y_n)_n$, $(a_n)_n$, and $(b_n)_n$ of rationals converging to $x, y, \beta, \beta$ respectively such that

$$\lim_n \frac{|x - x_n|}{|\beta - a_n| + 2^{-n}} = 0, \ \lim_n \frac{|y - y_n|}{|\beta - b_n| + 2^{-n}} = 0. \tag{3}$$

For each $k \in \omega$, pick up $n = n(k) > k$ such that

$$|a_n - b_n| < 2^{-k-1}, \tag{4}$$

which should exist because $a_n, b_n \to \beta$. Then,

$$|\beta - a_{n(k)}| + 2^{-k} \geq |\beta - a_{n(k)}| + 2^{-n(k)}$$

and

$$\begin{aligned} |\beta - a_{n(k)}| + 2^{-k} &\geq |\beta - b_{n(k)}| - |a_{n(k)} - b_{n(k)}| + 2^{-k} \\ &> |\beta - b_{n(k)}| - 2^{-k-1} + 2^{-k} \\ &\geq |\beta - b_{n(k)}| + 2^{-n(k)}, \end{aligned} \tag{5}$$

which implies

$$\frac{|f(x,y) - f(x_{n(k)}, y_{n(k)})|}{|\beta - a_{n(k)}| + 2^{-k}} \leq L \frac{|x - x_{n(k)}|}{|\beta - a_{n(k)}| + 2^{-n(k)}} + L \frac{|y - y_{n(k)}|}{|\beta - b_{n(k)}| + 2^{-n(k)}} \to 0,$$

where $L$ is a Lipschitz constant on a neighborhood of $(x, y)$. Then, $(f(x_{n(k)}, y_{n(k)}))_k$ and $(a_{n(k)})_k$ are the desired sequences. Hence, we have $f(x, y) \ll_S \beta$. □

## 4. Quasi Solovay reducibility

In our earlier research [10, 9], we proposed the concept of quasi-Solovay reducibility specifically concerning left-c.e. reals. Here, we aim to expand this concept to computably approximable reals.

### 4.1. Definition and coincidence.

**Notation 4.1.** For $\alpha \in \mathbf{CA}$, let $\mathrm{CS}(\alpha)$ denote the set of all computable sequences of rationals converging to $\alpha$. For $\alpha \in \mathbf{LC}$, let $\mathrm{ICS}(\alpha)$ denote the set of all increasing computable sequences of rationals converging to $\alpha$. Let $\mathbf{EC}_{>0}$ be the set of positive computable reals.

We use the following characterization of quasi Solovay reducibility for left-c.e. reals [10, Lemma 3.1(4)] where we use $\mathbf{EC}_{>0}$ rather than positive integers.

**Definition 4.2.** Let $\alpha, \beta \in \mathbf{LC}$. We say that $\alpha$ is quasi Solovay reducible to $\beta$, denoted by $\alpha \leq_{qS} \beta$, if there are $(a_n)_n \in \mathrm{ICS}(\alpha)$, $(b_n)_n \in \mathrm{ICS}(\beta)$, and $s, q \in \mathbf{EC}_{>0}$ such that

$$\alpha - a_n \leq q(\beta - b_n)^s$$

for all $n \in \omega$.

We extend this notion to computably approximable reals.

**Definition 4.3.** Let $\alpha, \beta \in \mathbf{CA}$. Let $s, q \in \mathbf{EC}_{>0}$. We say that $(\alpha, \beta)$ satisfies $(s, q)$-*Solovay relation via* $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$ if

$$|\alpha - a_n| \leq q(|\beta - b_n|^s + 2^{-n}) \tag{6}$$

for all $n \in \omega$.

**Observation 4.4.** *Let $\alpha, \beta \in \mathbf{CA}$. Then, $\alpha \leq_S \beta$ if and only if $(\alpha, \beta)$ satisfies $(1, q)$-Solovay relation via some sequences for some $q \in \omega$.*

**Proposition 4.5.** *Let $\alpha, \beta \in \mathbf{LC}$. Then, $\alpha \leq_{qS} \beta$ if and only if $(\alpha, \beta)$ satisfies $(s, q)$-Solovay relation via some $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$ for some $s, q \in \mathbf{EC}_{>0}$.*

*Proof.* The "only if" direction is immediate.

We give a proof of the "if" direction by modifying a proof of a similar result in Zheng and Rettinger [24, Theorem 5]. Suppose that $(\alpha,\ \beta)$ satisfies $(s, q)$-Solovay relation via $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$ for some $s, q \in \mathbf{EC}_{>0}$. We can assume $0 < s \leq 1$.

**Claim:** We can further assume that $(a_n)_n \in \mathrm{ICS}(\alpha)$.

If $a_n < \alpha$ at most finitely many $n$, then $\alpha$ is right-c.e., which implies that $\alpha$ is computable and the claim is obvious.

If $a_n < \alpha$ for infinitely many $n$, then fix $(c_n)_n \in \mathrm{ICS}(\alpha)$ by $\alpha \in \mathbf{LC}$. We take a sub-sequence $(a_{n_i})_i \in \mathrm{ICS}(\alpha)$ as follows. Let $n_0$ be such that $a_{n_0} < \alpha$. For a defined $a_{n_i} < \alpha$ by induction hypothesis, we can computably find $n > n_i$ and $m$ such that

$$a_{n_i} < a_n < c_m,$$

and let $n_{i+1}$ be this $n$. Note that $(\alpha,\ \beta)$ satisfies $(s, q)$-Solovay relation via $(a_{n_i})_i \in \mathrm{ICS}(\alpha)$ and $(b_{n_i})_i \in \mathrm{CS}(\beta)$. This is the end of the proof of the claim.

The same argument applies to $(b_n)_n$ and we also assume that $(b_n)_n$ is increasing.

Now, we use the power mean inequality:

$$\left(\frac{x^s + y^s}{2}\right)^{1/s} \leq \frac{x + y}{2} \text{ for any } x, y \geq 0, \text{ and } 0 < s \leq 1. \tag{7}$$

When we substitute $\beta - b_n$ and $2^{-n/s}$ for $x$ and $y$, we obtain

$$\alpha - a_n \leq q((\beta - b_n)^s + 2^{-n}) \leq 2^{1-s} q(\beta - b_n + 2^{-n/s})^s.$$

Here, $(b_n)_n$ and $(-2^{-n/s})_n$ are increasing and the sequence $(b_n - 2^{-n/s})_n$ of the sums is an increasing computable sequence of rationals converging to $\beta$. Hence, $\alpha \leq_{qS} \beta$. □

Now, we call this notion quasi Solovay reducibility for computably approximable reals.

**Definition 4.6.** Let $\alpha, \beta \in \mathbf{CA}$. Then, $\alpha$ is quasi Solovay reducible to $\beta$, denoted by $\alpha \leq_{qS} \beta$, if $(\alpha, \beta)$ satisfies $(s, q)$-Solovay relation for some $s, q \in \mathbf{EC}_{>0}$.

4.2. **Some characterizations.** As done for Solovay reducibility in the previous paper [11], we give a Cauchy-style characterization and a characterization via Hölder continuous functions for quasi-Solovay reducibility. Most key observations have already been done in the previous work.

**Proposition 4.7** (See [11, Proposition 3.5])**.** *Let $\alpha, \beta \in \mathbf{CA}$. Then, $\alpha \leq_{qS} \beta$ if and only if there are $(a_n)_n \in \mathrm{CS}(\alpha)$, $(b_n)_n \in \mathrm{CS}(\beta)$, and $s, q \in \mathbf{EC}_{>0}$ such that*

$$(\forall k, n \in \omega)[k < n \Rightarrow |a_n - a_k| < q(|b_n - b_k|^s + 2^{-k}).$$

Sketch of a proof: The case when $q = 1$ is shown by applying [11, Lemma 3.4] with $F(x) = x^s$. The general case reduces to the above case by considering $\alpha/q$ instead of $\alpha$.

**Proposition 4.8** (See [11, Theorem 3.7])**.** *Let $\alpha, \beta \in \mathbf{CA}$. Then, $\alpha \leq_{qS} \beta$ if and only if there are semi-comptable function interval $(f, h)$ such that*

(i) *$f, h$ are both $s$-Hölder continuous functions for some $0 < s \leq 1$,*
(ii) *$f(\beta) = h(\beta) = \alpha$.*

**Remark 4.9.** For a proof, apply [11, Lemma 3.8, 3.9] with $F(x) = x^s$ for $x \geq 0$. Notice that $F(x)$ is subadditive for $0 < s \leq 1$, that is,

$$(x + y)^s \leq x^s + y^s \text{ for } x, y \geq 0 \text{ and } 0 < s \leq 1. \tag{8}$$

This is because

$$1 = \frac{x}{x+y} + \frac{y}{x+y} \leq \left(\frac{x}{x+y}\right)^s + \left(\frac{y}{x+y}\right)^s$$

unless $x = y = 0$.

4.3. **Real closed field.** We showed that $S(\leq \beta)$ forms a real closed field in Theorem 3.1. We give a version of quasi Solovay reducibility.

**Proposition 4.10.** *Let $\beta$ be a computably approximable real. Then, the set of reals that are quasi-Solovay reducible to $\beta$ forms a real closed field.*

*Proof.* The proof is by straightforward modification of that of Theorem 3.13. We replace the equation (3) with the inequalities

$$\frac{|x - x_n|}{|\beta - a_n|^s + 2^{-n}} \leq q, \quad \frac{|y - y_n|}{|\beta - b_n|^s + 2^{-n}} \leq q,$$

for some $q \in \omega$. We also replace the inequality (4) with

$$|a_n - b_n| < 2^{-(k+1)/s}.$$

Then, instead of the inequality (5), we can deduce

$$\begin{aligned}|\beta - a_{n(k)}|^s + 2^{-k} &\geq |\beta - b_{n(k)}|^s - |a_{n(k)} - b_{n(k)}|^s + 2^{-k} \\ &\geq |\beta - b_{n(k)}|^s - 2^{-k-1} + 2^{-k} \\ &\geq |\beta - b_{n(k)}|^s + 2^{-n(k)},\end{aligned}$$

where we use that the function $x \mapsto x^s$ is subadditive for $0 < s \leq 1$ (8) for the first inequality. This implies that

$$\frac{|f(x,y) - f(x_{n(k)}, y_{n(k)})|}{|\beta - a_{n(k)}|^s + 2^{-k}} \leq L\frac{|x - x_{n(k)}|}{|\beta - a_{n(k)}|^s + 2^{-n(k)}} + L\frac{|y - y_{n(k)}|}{|\beta - b_{n(k)}|^s + 2^{-n(k)}} \leq 2qL.$$

□

## 5. Solovay and strong Solovay reducibility

In our previous work [11, Theorem 3.7], we characterized Solovay reducibility via Lipschitz continuous functions. In Proposition 4.8 we characterized quasi Solovay reducibility via Hölder continuous functions. The goal of this section is to characterize strong Solovay reducibility using derivatives.

5.1. **Eliminating the remainder term.** First, recall the definition of Solovay reducibility from Definition 2.1: For $\alpha, \beta \in \mathbf{CA}$, $\alpha \leq_S \beta$ if there are $(a_n)_n \in \mathrm{CS}(\alpha)$, $(b_n)_n \in \mathrm{CS}(\beta)$, and a positive integer $q \in \omega$ such that

$$|\alpha - a_n| < q(|\beta - b_n| + 2^{-n})$$

for all $n \in \omega$. Rettinger and Zheng [24, Section 2] observed that the relation without the remainder term $2^{-n}$ is a different notion and does not behave well. However, we can characterize Solovay reducibility without the remainder term by considering limits as follows.

**Theorem 5.1.** *Let $\alpha, \beta \in \mathbf{CA}$. Then, $\alpha \leq_S \beta$ if and only if there exist sequences $(a_n)_n \in \mathrm{CS}(\alpha)$, $(b_n)_n \in \mathrm{CS}(\beta)$, a continuous function $f : \mathbb{R} \to \mathbb{R}$, and a constant $q \in \omega$ such that*

(i) *the slope $\frac{f(x)-f(\beta)}{x-\beta}$ for $x \neq \beta$ is bounded by $q$, and*
(ii) *$|f(b_n) - a_n| \leq 2^{-n}$ for all $n \in \omega$.*

*Furthermore, we can also impose that $f$ is lower semicomputable and Lipschitz continuous with Lipschitz constant $q$.*

*Proof.* ("if" direction) By item (ii), we have $f(\beta) = \alpha$. For each $n$ such that $b_n \neq \beta$, we have

$$|\alpha - a_n| \leq |f(\beta) - f(b_n)| + |f(b_n) - a_n| \leq q|\beta - b_n| + 2^{-n}$$

by item (i) and item (ii). These inequalities are also true for each $n$ such that $b_n = \beta$.

("only if" direction) Suppose that $\alpha \leq_S \beta$. By the Cauchy-style characterization [11, Proposition 3.5] for Solovay reducibility, there exist $(a_n)_n \in \mathrm{CS}(\alpha)$, $(b_n)_n \in \mathrm{CS}(\beta)$, and $q \in \omega$ such that

$$(\forall k, n)[k < n \Rightarrow |a_k - a_n| \leq q|b_k - b_n| + 2^{-k}]. \tag{9}$$

In the proof of a result in [11, Lemma 3.9], we showed that the function

$$f(x) = \sup_{n\in\omega}(a_n - q|x - b_n| - 2^{-n}) \tag{10}$$

has the following properties:

(a) $f$ is lower semicomputable.
(b) $f$ is Lipschitz continuous with Lipschitz constant $q$.

The item (b) implies item (i) and that $f$ is continuous.

Now, it suffices to show item (ii). Fix $n$. By the definition of $f$ (10), we have

$$f(b_n) = \sup_{k\in\omega}(a_k - q|b_n - b_k| - 2^{-k}) \geq a_n - 2^{-n}$$

where we let $k = n$ for the last inequality. By assumption (9), we have

$$\begin{aligned} k < n \Rightarrow & a_k - q|b_k - b_n| - 2^{-k} \leq a_n, \\ n \leq k \Rightarrow & a_k - q|b_n - b_k| - 2^{-k} \leq a_n + 2^{-n} - 2^{-k}. \end{aligned}$$

These facts imply $f(b_n) \leq a_n + 2^{-n}$. Hence, item (ii) is proved. □

5.2. **Characterization using derivatives.** We provide a characterization for strong Solovay reducibility using derivatives.

Recall from Definition 3.8 that, for $\alpha, \beta \in \mathbf{CA}$, $\alpha \ll_S \beta$ if there are $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$ such that $\lim_{n\to\infty} \dfrac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}} = 0$.

**Theorem 5.2.** *Let $\alpha, \beta \in \mathbf{CA}$. Then, $\alpha \ll_S \beta$ if and only if there exist $(a_n)_n \in \mathrm{CS}(\alpha)$, $(b_n)_n \in \mathrm{CS}(\beta)$, and a continuous function $g$ such that*

(i) *the derivative $g'(\beta) = 0$,*
(ii) *$|g(b_n) - a_n| \leq 2^{-n}$ for all $n$.*

*We can further impose that $g$ is differentiable on the real line.*

Notice that the condition is stronger than that in Theorem 5.1.

**Remark 5.3.** The function $g$ need not be computable, but $g(b_n)$ should be close to $a_n$ because of the condition (ii) and the sequences $(a_n)_n$ and $(b_n)_n$ should be computable.

**Remark 5.4.** We can not impose the following condition: For every $L > 0$, $g$ is an $L$-Lipschitz function. This would imply that $g$ is a constant function, which is possible only when $\alpha$ is computable.

We do not know whether we can impose that $g \in C^1$.

*Proof.* ("if" direction)

Let $p(n) = n^2$ for $n \in \omega$. Let $k, n \in \omega$ be such that $p(n) \leq k$. By the triangle inequality, we have

$$|a_k - a_{p(n)}| \leq |a_k - g(b_k)| + |g(b_k) - g(b_{p(n)})| + |g(b_{p(n)}) - a_{p(n)}|.$$

By letting $k \to \infty$, we have

$$|\alpha - a_{p(n)}| \leq |g(\beta) - g(b_{p(n)})| + 2^{-p(n)} \tag{11}$$

for all $n \in \omega$.

Fix $\epsilon > 0$. Since $g'(\beta) = 0$, there exists $\delta > 0$ such that

$$0 < |x - \beta| < \delta \Rightarrow \left|\frac{g(x) - g(\beta)}{x - \beta}\right| < \epsilon.$$

By $b_n \to \beta$, there exists $N \in \omega$ such that

$$|b_{p(n)} - \beta| < \delta$$

for all $n \geq N$. Then, for such $n \geq N$, we have

$$|g(b_{p(n)}) - g(\beta)| < \epsilon |b_{p(n)} - \beta|. \tag{12}$$

By the inequalities (11) and (12), we have

$$\frac{|\alpha - a_{p(n)}|}{|\beta - b_{p(n)}| + 2^{-n}} \leq \frac{\epsilon|\beta - b_{p(n)}| + 2^{-p(n)}}{|\beta - b_{p(n)}| + 2^{-n}} \leq \epsilon + 2^{-p(n)+n}.$$

This implies $\alpha \ll_S \beta$.

("only if" direction)

Suppose that $\alpha \ll_S \beta$ via $(a_n)_n \in \mathrm{CS}(\alpha)$ and $(b_n)_n \in \mathrm{CS}(\beta)$. We also assume that

$$\frac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}} \leq \frac{1}{2}. \tag{13}$$

We can assume $\beta$ is not a rational and $b_n \neq b_k$ for each distinct $n, k \in \omega$ by making minor adjustments as required. We construct a continuous function $g$ such that the conditions (i) and (ii) hold.

We only construct a continuous function $g(x)$ for $x \geq \beta$. The construction of $g(x)$ for $x \leq \beta$ is similar. We use the following sets of indices of $(a_n)_n$ and $(b_n)_n$:

$$A = \{n \in \omega \ : \ \beta < b_n, \ |\beta - b_n| \leq 2^{-n}\},$$
$$B = \{n \in \omega \ : \ \beta < b_n, \ |\beta - b_n| > 2^{-n}\}.$$

Then, connect the following points with a poly-line:

$$P = \{(b_n, \ \alpha) \ : \ n \in A\} \cup \{(b_n, \ a_n) \ : \ n \in B\}.$$

If $A \cup B$ is the empty set, then the following $g$ satisfies the desired property: $g(x) = \alpha$ for all $x \geq \beta$. If $A \cup B$ is a non-empty finite set, then connect $(\beta, \alpha)$ and the left-most point of $P$ so that the derivative at $\beta$ is 0. Such a poly-line is well-defined since we have assumed $b_n \neq b_k$ for each distinct $n, k \in \omega$. Clearly, $g$ is continuous.

Hereafter, we assume that $A \cup B$ is an infinite set. We claim that the right derivative of $g$ at $\beta$ is 0. Fix $\epsilon > 0$. Then, there exists $N \in \omega$ such that, for all $n \geq N$, we have

$$|\alpha - a_n| \leq \epsilon(|\beta - b_n| + 2^{-n}).$$

If $n \in A$, we have $|\alpha - g(b_n)| = 0 \leq 2\epsilon|\beta - b_n|$. If $n \in B$ and $n \geq N$, then we have

$$|\alpha - g(b_n)| = |\alpha - a_n| \leq 2\epsilon|\beta - b_n|.$$

Thus, all points $(x, y) = (b_n, g(b_n))$ for $n \geq N$ of the poly-line satisfies

$$|\alpha - y| \leq 2\epsilon|\beta - x|,$$

which is equivalent to

$$\alpha - 2\epsilon(x - \beta) \leq y \leq \alpha + 2\epsilon(x - \beta),$$

and so does every point $(x, g(x))$ because the region represented by this inequality is convex. Let

$$\delta = \min\{|\beta - b_n| \ : \ n < N\} > 0.$$

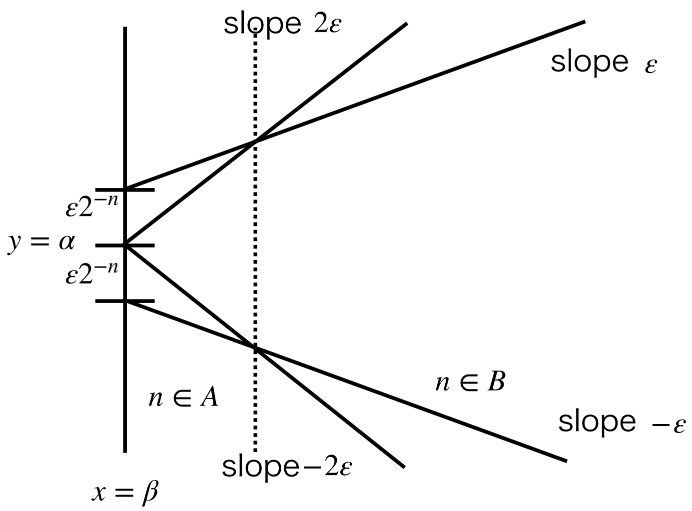


FIGURE 1. possible positions of $(x, g(x))$

Then, for each $x$ such that $\beta < x < \beta + \delta$, we have

$$\left|\frac{g(x) - g(\beta)}{x - \beta}\right| \leq 2\epsilon,$$

which implies that the right derivative of $g$ at $\beta$ is 0.

We show that $|g(b_n) - a_n| \leq 2^{-n}$ for all $n$. For each $n \in B$, this is true because $g(b_n) = a_n$. For each $n \in A$, we have

$$|g(b_n) - a_n| = |\alpha - a_n| \leq \frac{1}{2}(|\beta - b_n| + 2^{-n}) \leq 2^{-n}.$$

(differentiability)

Next, we modify $g$ to create a differentiable function $\hat{g}$. Focus on a line segment of $g$ connecting $(x_0, y_0)$ and $(x_1, y_1)$.

We use the function

$$f(x) = \begin{cases} \exp(-1/x) & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

which is non-negative and infinitely differentiable. This fact is particularly well-known in the field of analysis and can be verified through simple calculations.

The function $f$ is used to construct a smooth transition function:

$$h_0(x) = \frac{f(x)}{f(x) + f(1 - x)},$$

which is infinitely differentiable, $h_0(x) = 0$ for $x \leq 0$ and $h_0(x) = 1$ for $x \geq 1$. Thus, $h_0$ is that smoothly transitions from the constant function $x \mapsto 0$ to the constant function $x \mapsto 1$ over the interval $[0, 1]$.

Here, we construct a function that smoothly transitions from a constant function to a line with a possibly non-zero slope. Let $y = ax + b$ be the equation of the line passing through two points $(x_0, y_0)$ and $(x_1, y_1)$. Let $\delta > 0$ be a sufficiently small rational. We consider the functions

$$h_1(x) = \frac{(ax+b)f(x-x_0) + y_0 f(x_0+\delta-x)}{f(x-x_0)+f(x_0+\delta-x)} \text{ for } x \in \mathbb{R},$$

and

$$h_2(x) = \frac{y_1 f(x-(x_1-\delta)) + (ax+b)f(x_1-x)}{f(x-(x_1-\delta))+f(x_1-x)} \text{ for } x \in \mathbb{R}.$$

Both functions are infinitely differentiable. Furthermore, $h_1(x) = y_0$ for $x \leq x_0$, $h_1(x) = ax+b$ for $x \geq x_0+\delta$, and $h_1(x)$ is strictly increasing for $x_0 \leq x \leq x_0+\delta$. The function $h_2$ has similar properties.

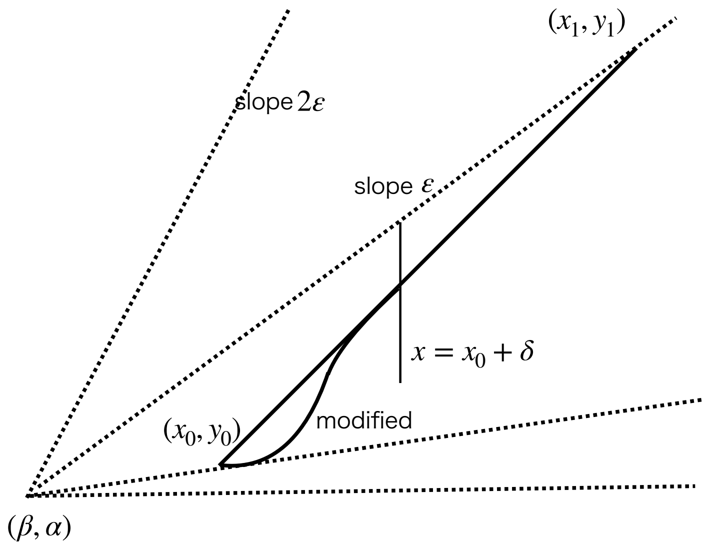


FIGURE 2. modification of $g$

Then, we define $\hat{g}(x)$ for $x_0 \leq x \leq x_1$ as follows:

$$\hat{g}(x) = \begin{cases} h_1(x) & (x_0 \leq x \leq x_0+\delta) \\ ax+b & (x_0+\delta \leq x \leq x_1-\delta) \\ h_2(x) & (x_1-\delta \leq x \leq x_1). \end{cases}$$

We are going to show $\hat{g}(x)$ is infinitely differentiable on this interval. Since $h_1(x)$ is infinitely differentiable and $h_1(x)$ is a constant function for $x \leq x_0$, the $n$-th derivative of $h_1$ at $x_0$ is 0 for every $n \geq 1$. Hence, the right $n$-th derivative of $\hat{g}(x)$ at $x_0$ is 0. Similarly, the left $n$-th derivative at $x_1$ is 0 for every $n \geq 1$.

We do this modification for each line segment of $g$. Then, $g$ is infinitely differentiable for all $x \neq \beta$.

To claim that $\hat{g}$ is differentiable, it suffices to show that the first derivative of $\hat{g}$ at $x = \beta$ is 0. Consider the absolute values of the slope of the line segment connecting $(\beta, \alpha)$ and $(x, \hat{g}(x))$ and those connecting $(\beta, \alpha)$ and $(x, g(x))$. Define

$$\hat{m} = \max\{\left|\frac{\hat{g}(x)-\alpha}{x-\beta}\right| \ : \ x_0 \leq x \leq x_1\}, \ m = \max\{\left|\frac{g(x)-\alpha}{x-\beta}\right| \ : \ x_0 \leq x \leq x_1\}.$$

Then, $\hat{m}$ goes to $m$ if the value $\delta$ in the definition $\hat{g}$ goes to 0. Thus, we can take the value $\delta$ sufficiently small so that

$$\hat{m} \leq 2m.$$

Notice that $\delta$ depends on the line segment. Since the right-hand side goes to 0 as $x \to \beta$, so does the left-hand side. □

## 6. Strong Solovay and strong K-reducibility

This section investigates the relationship between two Solovay reducibilities and two $K$-reducibilities. Here, we identify a real in $[0,1]$ and its infinite binary expansion. For two reals $\alpha, \beta \in [0,1]$, $\alpha$ is $K$-reducible to $\beta$, denoted by $\alpha \leq_K \beta$, if $K(\alpha \upharpoonright n) \leq K(\beta \upharpoonright n) + O(1)$. The real $\alpha$ is strongly $K$-reducible to $\beta$, denoted by $\alpha \ll_K \beta$, if $\lim_{n\to\infty}(K(\beta \upharpoonright n) - K(\alpha \upharpoonright n)) = \infty$.

The goal of this section is to show the strict implication

$$\ll_K \Rightarrow \ll_S \Rightarrow \leq_S \Rightarrow \leq_K$$

for left-c.e. reals, and that this does not hold for weakly computable reals. One can see that the two Solovay reducibilities are sandwiched between the two $K$-reducibilities.

### 6.1. The case for left-c.e. reals.

First, we will see that strong Solovay reducibility is sandwiched between strong $K$-reducibility and Solovay reducibility.

**Proposition 6.1.** *Let $\alpha$ and $\beta$ be left-c.e. reals in $[0,1]$. If $\alpha \ll_K \beta$, then $\alpha \ll_S \beta$.*

The proof is based on [14, Proposition 2.3].

*Proof.* Let $(a_s)_s \in \mathrm{ICS}(\alpha)$ and $(b_s)_s \in \mathrm{ICS}(\beta)$. Given $n \in \omega$, let $s_n$ be the first stage $s$ such that

$$a_s \upharpoonright n = \alpha \upharpoonright n.$$

Then, there exists a constant $c_1 \in \omega$ such that, for each $n \in \omega$,

$$K(b_{s_n} \upharpoonright n) \leq K(a_{s_n} \upharpoonright n) + c_1 = K(\alpha \upharpoonright n) + c_1. \tag{14}$$

This is because, given $a_{s_n} \upharpoonright n$, one can compute the stage $s_n$ from the approximation $(a_s)_s$, which also computes $b_{s_n} \upharpoonright n$.

For a string $\sigma$ of length $n$, its lexicographic rank, denoted $r(\sigma)$, is its position when all $n$-bit strings are sorted lexicographically. This rank is a natural number between 1 and $2^n$. For instance, $r(0^n) = 1$ and $r(1^n) = 2^n$. Let

$$d_n = r(\beta \upharpoonright n) - r(b_{s_n} \upharpoonright n).$$

Then, $\beta \upharpoonright n$ can be computed from $b_{s_n} \upharpoonright n$ and $d_n$. Thus, there exists a constant $c_2 \in \omega$ such that, for each $n \in \omega$,

$$K(\beta \upharpoonright n) \leq K(b_{s_n} \upharpoonright n) + K(d_n) + c_2. \tag{15}$$

By inequalities (14) and (15) with the assumption $\alpha \ll_K \beta$, we have $d_n \to \infty$ as $n \to \infty$. If $n$ is sufficiently large and $s_n \leq s \leq s_{n+1}$, then

$$\alpha - a_s \leq \alpha - a_{s_n} \leq 2^{-n},$$
$$\beta - b_s \geq \beta - b_{s_{n+1}} \geq (d_{n+1} - 1)2^{-n-1}.$$

Thus,

$$\frac{\alpha - a_s}{\beta - b_s} \leq \frac{2}{d_{n+1} - 1} \to 0$$

as $s \to \infty$. Hence, $\alpha \ll_S \beta$. □

The implication in Proposition 6.1 does not reverse. This is because there exists a non-ML-random left-c.e. real $\alpha$ such that $\alpha \not\ll_K \Omega$, which will be shown in Proposition 6.2 below. Since $\alpha \ll_S \Omega$, this pair is a counterexample.

**Proposition 6.2.** *There exists a left-c.e. real $\alpha$ such that $\alpha$ is not ML-random and $\alpha \not\ll_K \Omega$.*

*Proof.* Barmpalias, Downey, and Greenberg [1] showed that the following are equivalent for a c.e. degree $\mathbf{d}$.

(i) There is a left-c.e. real $\alpha \leq_T \mathbf{d}$ not cl-reducible to any ML-random left-c.e. real.
(ii) $\mathbf{d}$ is array noncomputable.

Notice that such a degree exists. Let $\alpha$ be such a real. Then, $\alpha$ is not ML-random.

Let $\Omega$ be a left-c.e. ML-random real. Since $\alpha$ is left-c.e., we have $\alpha \leq_S \Omega$ and $K(\alpha \restriction n) \leq K(\Omega \restriction n) + O(1)$. If $\beta \ll_K \gamma$ for left-c.e. reals $\beta, \gamma$, then $\beta <_{cl} \gamma$ [5, Theorem 9.12.1]. Thus, if $\alpha \ll_K \Omega$, then $\alpha \leq_{cl} \Omega$, which is a contradiction. □

For left-c.e. reals $\alpha, \beta$, if $\alpha \ll_S \beta$, then $\alpha \leq_S \beta$ by definition. This implication does not reverse by Proposition 3.11.

6.2. **The case for weakly computable reals.** The implications we have shown above for left-c.e. reals does not hold for weakly computable reals.

**Proposition 6.3.** *There exist $\alpha, \beta \in$ WC such that $\alpha \ll_K \beta$ and $\alpha \not\leq_S \beta$.*

*Proof.* Consider Tadaki's Omega $\Omega^d = \sum_{\sigma \in \mathrm{dom}(U)} 2^{-\frac{|\sigma|}{d}}$ for a computable real $d \in (0,1)$. Then, we have

$$dn - O(1) < K(\Omega^d \restriction n) < dn + o(n)$$

(see [23, Theorem 3.2]). We set $\beta = \Omega^{1/2}$. Then, $\beta$ is a left-c.e. real such that $\frac{1}{3}n < K(\beta \restriction n) < \frac{2}{3}n$ for sufficiently large $n$, say, for all $n \geq N$. Fix an increasing computable sequence $(\beta_s)_s$ of rationals converging to $\beta$. For each $n \geq N$, we have

$$K(\beta_s \restriction n) < \frac{2}{3}n$$

for all sufficiently large $s$. By taking a sub-sequence, we can assume that $(\beta_s)_s$ satisfies this inequality for all $n$ such that $N \leq n \leq s$. Then, the number of candidates $\beta_s \restriction n$ is at most $2^{2n/3}$.

We construct a real $\alpha \in$ WC such that $\alpha \ll_K \beta$ and $\alpha \not\leq_S \beta$. At stage $s$, we will define a finite string $\sigma_s = \sigma = \sigma(1)\sigma(2)\cdots\sigma(|\sigma|)$ corresponding the real $\sum_{\sigma(k)=1, 1 \leq k \leq |\sigma|} 2^{-k}$, and define $\alpha = \lim_n \sigma_n$. We put the same number in each of the consecutive four digits, that is,

$$\sigma(4m+1) = \sigma(4m+2) = \sigma(4m+3) = \sigma(4m+4) \text{ for each } m \in \omega. \tag{16}$$

This implies $K(\alpha \restriction n) < \frac{n}{4} + K(n) + O(1)$, which ensures $\alpha \ll_K \beta$.

To ensure $\alpha \not\leq_S \beta$, we use the following characterization of Solovay reducibility [24, Definition 3, Theorem 6]. For $x, y \in \mathrm{WC}$, $x \leq_S y$ if and only if for any computable sequence $(x_s)_s$ and $(y_s)_s$ of rationals converging to $x, y$ respectively, there are an increasing computable functions $f$ and a constant $c$ such that $(\forall s)(|x - x_{f(s)}| \leq c(|y - y_{f(s)}| + 2^{-s}))$. Thus, if $x \in \mathrm{WC}$, $y \in \mathrm{LC}$, $(y_s)_s$ is a computable increasing sequence converging to $y$, and $x \leq_S y$, then there exist a computable function $g : \omega \to \mathbb{Q}_2$, a computable increasing function $f : \omega \to \omega$, and a constant $c$ such that $(\forall s)(|x - g(s)| \leq 2^c(y - y_{f(s)} + 2^{-s}))$, where $\mathbb{Q}_2$ is the set of all dyadic rationals.

We have already fixed the sequence $(\beta_s)_s$ converging to $\beta$. Fix a computable enumeration of all triples $(g_k, f_k, c_k)_k$ satisfying the following properties (i)-(iv).

(i) $g_k : \omega \to \mathbb{Q}_2$ is a partial computable function,
(ii) $f_k : \omega \to \omega$ is a partial computable function,
(iii) if $f_k(s)\downarrow$, then $f_k(s-1)\downarrow$ and $f_k(s-1) < f_k(s)$ for all $k$ and $s$,
(iv) $c_k \in \omega$.

Here, we may assume that each triple satisfying (i)-(iv) appears infinitely many times in the enumeration.

We try to ensure that $(g_k, f_k, c_k)$ is not a witness of Solovay reducibility. We use a computable increasing function $h : \omega \to \omega$ specified later. Let

$$v(k, s) = \min\{t \leq s \ : \ \beta_t \upharpoonright h(k) = \beta_s \upharpoonright h(k)\}.$$

For each $k$, $v(k, s)$ is increasing in $s$ and convergent. Let $v(k) = \lim_s v(k, s)$. If $v(k) < h(k)$ for infinitely many $k$, then we can compute $\beta \upharpoonright h(k) = \beta_{h(k)} \upharpoonright h(k)$ from $k$ and $K(\beta \upharpoonright h(k)) < K(k) + O(1)$, which contradicts $K(\beta \upharpoonright n) > \frac{1}{3}n$ for sufficiently large $n$ (provided that $h$ grows fast enough). Hence, we can assume that

$$v(k) \geq h(k) \text{ for all } k \geq N. \tag{17}$$

We use the following requirement:

$$R_k : (\exists t \geq v(k))[g_k(t)\uparrow \vee f_k(t)\uparrow \vee |\alpha - g_k(t)| > 2^{-h(k)+c_k+1}].$$

Since each triple satisfying (i)-(iv) appears infinitely many times in the numeration, it is sufficeint to satisfy $R_k$ for all $k \geq N$. In the following, we consider $R_k$ only for $k \geq N$.

At stage $s = 0$, each $R_k$ has not met. At stage $s \geq h(k)$, if $g_k(t)[s]\downarrow$ and $f_k(t)[s]\downarrow$ for some $t$ such that $v(k, s) \leq t \leq s$, then $R_k$ requires attention. We will construct $\sigma_s$ as a prefix of $\alpha$ so that $|\alpha - g_k(t)| > 2^{-h(k)+c_k+1}$ assuming $s \geq v(k) = v(k, s)$, by which we will show $\beta - \beta_{f_k(t)} \leq \beta - \beta_t < 2^{-h(k)}$. After the stage, one may find an increase of $v(k, s)$. In such a case, $R_k$ is no longer met. Notice that each requirement $R_k$ will be injured at most finitely many times. We will count the number of injuries later.

We now give the construction of $\sigma_s$. The $\sigma_0$ is the empty string. For each stage $s$, we try to meet the requirement $R_k$ that requires attention. We temporarily extend $\sigma_{s-1}$ so that for each $i \leq h(s-1)$, if $i$ is not in the domain of $\sigma_{s-1}$ then $\sigma_{s-1}(i) = 0$. For each $k \leq s$, we may change $(h(k-1)+1)$-th to $h(k)$-th bits of $\sigma_{s-1}$ so we can do this in parallel. Let $\ell$ be the smallest multiple of 4 greater than $c_k + 1$. We may assume that $h(k-1) + 1 \leq h(k) - \ell$. If

$g_k(s)$ satisfies property (16) with $h(k-1)+1$-th to $h(k)-\ell$-th bits and $g_k(s)$ and $\sigma_{s-1}$ have the same $(h(k)-\ell-3)$-th to $(h(k)-\ell)$-th bits, then we define $\sigma_s$ by modifying $\sigma_{s-1}$ so that

- $\sigma_s$ satisfies property (16) with $(h(k-1)+1)$-th to $h(k)$-th bits.
- at least one digit between the $(h(k)-\ell-3)$-th and $(h(k)-\ell)$-th digits of the strings $\sigma_s$ and $g_k(s)$ is different,

Otherwise, let $\sigma_s$ be the string $\sigma_{s-1}$ modified above. Finally, we set $\alpha = \lim_s \sigma_s$.

The relation $\alpha \ll_K \beta$ is immediate from the construction, the reason for which was already explained above.

We claim that $\alpha \not\leq_S \beta$. Suppose that, for some $k$, we have

$$|\alpha - g_k(s)| \leq 2^{c_k}(\beta - \beta_{f_k(s)} + 2^{-s}) \tag{18}$$

for all $s$. Since $v(k)$ will stabilize eventually, there exist $s, t$ such that

$$s \geq h(k),\ \ v(k,s) = v(k) \leq t \leq s,\ \ g_k(t)[s]\downarrow,\ \text{and}\ f_k(t)[s]\downarrow.$$

Since $\sigma_s$ and $g_k(t)$ differ in at least one of the $(h(k-1)+1)$-th through $(h(k)-\ell)$-th bits, we have

$$|\alpha - g_k(t)| > 2^{-h(k)+\ell} > 2^{-h(k)+c_k+1}. \tag{19}$$

In contrast, since $v(k) \leq t$, we have $\beta_t \upharpoonright h(k) = \beta \upharpoonright h(k)$ and

$$\beta - \beta_{f_k(t)} \leq \beta - \beta_t < 2^{-h(k)},$$

which implies

$$2^{c_k}(\beta - \beta_{f_k(t)} + 2^{-t}) \leq 2^{-h(k)+c_k} + 2^{-v(k)+c_k} \leq 2^{-h(k)+c_k+1}, \tag{20}$$

where the second inequality follows from the inequality (17). However, the inequalities (19) and (20) contradict with the assumption (18).

Now we show that $\alpha$ is weakly computable by letting $h$ grow fast enough. When defining $\sigma_s$, the difference of two reals corresponding to $\sigma_{s-1}$ and $\sigma_s$ is at most $2^{-h(k)+c_k+O(1)}$ for each $k$. The number of the change of $v(k,s)$ is bounded by the number of the change of $\beta_s \upharpoonright h(k)$ (because $(\beta_s)_s$ is increasing), which is at most $2^{2h(k)/3}$. Thus, the total difference is, at most

$$\sum_k 2^{2h(k)/3} \times 2^{-h(k)+c_k+O(1)},$$

which is finite if $h$ grows fast enough. □

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