Quantifier Variations in Solovay Reducibility

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Topic

algorithmic randomness $\approx$ computability theory + probability theory

Solovay reducibility concerns real numbers, not binary sequences.

Solovay reducibility is induced by a partially computable version of Lipschitz continuous functions.

Thus, it relates both logic and analysis.

Goal

I claim that the following six are distinct concepts:

- $\forall x \forall y \, P(x, y)$
- $\forall x \exists y \, P(x, y)$
- $\forall y \exists x \, P(x, y)$
- $\exists x \forall y \, P(x, y)$
- $\exists y \forall x \, P(x, y)$
- $\exists x \exists y \, P(x, y)$

Interpret the predicate $P(x, y)$ as “x likes y.”

Goal

In the case of Solovay reducibility.

Summary

- Solovay reducibility for left-c.e. reals is robust.
- Solovay reducibility for c.a. reals is fragile

  $\rightarrow$ Construct a counterexample via the priority method
- Provide characterizations that are robust in both cases

Definition (left-c.e. case)

Definition

A real number α is **left-c.e.** (left-computably enumerable) if it is the limit of a computable increasing sequence of rationals.

Definition

For left-c.e. reals α, β, we say α is **Solovay reducible** to β if there exist computable increasing sequences $(a_s)_s$, $(b_s)_s$ and $q \in \omega$ such that

$$(\forall s \in \omega)\,[\alpha - a_s < q(\beta - b_s)]$$

holds.

Note: This is a commonly-used characterization, which differs slightly from the original definition by Solovay (1975).

Formalization (left-c.e. case)

Define

$$P_0^L : (\exists q \in \omega)(\forall s \in \omega)[\alpha - a_s < q(\beta - b_s)].$$

Then set:

- **(L-I)** $(\exists(a_s))(\exists(b_s))\, P_0^L$
- **(L-II)** $(\forall(b_s))(\exists(a_s))\, P_0^L$
- **(L-III)** $(\forall(a_s))(\exists(b_s))\, P_0^L$

Here $\exists$ and $\forall$ range over approximation sequences.

These (L-I), (L-II), and (L-III) are all equivalent. When adding a universal quantifier in the second place, one can take a computable subsequence to restore equivalence.

Definition (c.a. case)

Definition

A real number α is **c.a.** (computably approximable or Δ^0_2) if it is the limit of a computable sequence of rationals.
The sequence is not necessarily increasing.

Definition (Zheng and Rettinger 2004, S2a-reducibility)

For c.a. reals α, β, we say α is **Solovay reducible** to β if there exist computable approximations $(a_s)_s$, $(b_s)_s$ and $q \in \omega$ such that

$$(\forall s \in \omega)[|\alpha - a_s| < q\left(|\beta - b_s| + 2^{-s}\right)].$$

For left-c.e. reals, this coincides with the previous definition.

Formalization (c.a. case)

Define

$$P_0 : (\exists q \in \omega)(\forall s \in \omega)\left[|\alpha - a_s| < q\,(|\beta - b_s| + 2^{-s})\right].$$

Set:

- **(I)** $(\exists(a_s))(\exists(b_s))\,P_0$
- **(II)** $(\forall(b_s))(\exists(a_s))\,P_0$
- **(III)** $(\forall(a_s))(\exists(b_s))\,P_0$

Question

Are (I), (II), and (III) equivalent?

- Since they were robust for left-c.e. reals, one might expect the same for c.a. reals.
- Some papers assert this without proof.
- One quickly sees that the left-c.e. proof does not carry over, but constructing a counterexample is not easy.

Note: In (II), one may take a computable subsequence of (b_s); in (III), of (a_s), to recover equivalence with (I).

Main Result

Theorem

There exist left-c.e. reals α, β such that (I) holds but (II) does not.

Note: Don't be confused. This counterexample shows that definitions (I) and (II) for c.a. reals can be witnessed by left-c.e. reals. This does not contradict the fact that the two definitions coincide for left-c.e. reals.

Proof Sketch

Construct α, β using a finite injury priority argument.

To enforce (I), we construct increasing $(a_s)_s$, $(b_s)_s$ such that

$$\alpha - a_s \le \beta - b_s \text{ for all } s \in \omega$$

Negation of (II):

$$\exists (d_s)_s\ \forall (c_s)_s\ \forall q \in \omega\ \exists t \in \omega \left[|\alpha - c_t| \ge q \left(|\beta - d_t| + 2^{-t} \right) \right],$$

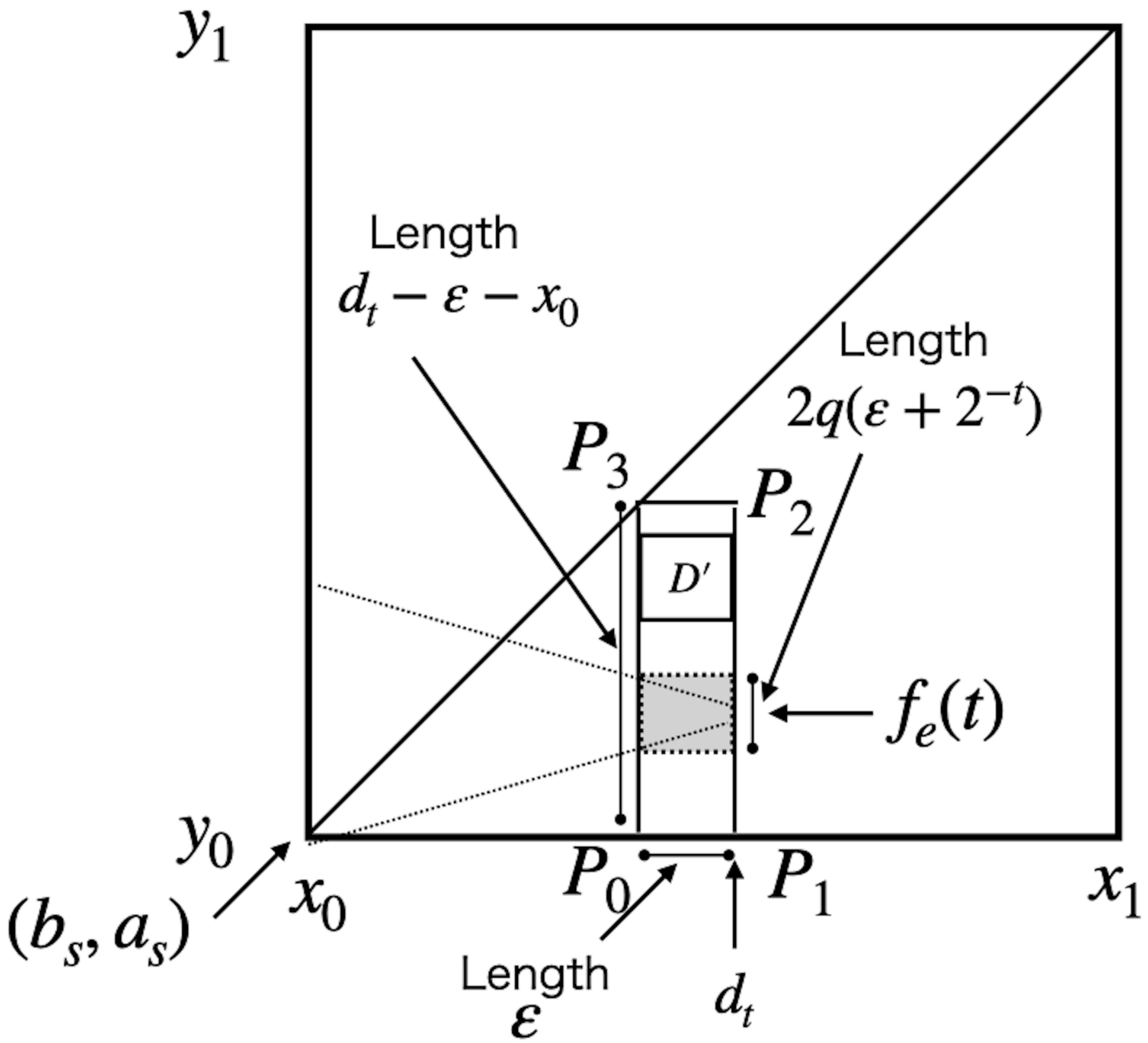
build an approximation $(d_s) \to \beta$ satisfying this condition.

Key idea

If β is close to b_s, then α should be close to a_s. If b_s is far from d_t, by letting β to be close to d_t, we have enough space to enforce both (I) and the negation of (II).

For simplicity, fix (c_s) and q.
Choose a large t. When c_t settles at stage $s > t$, place β near the current d_t.
An injury occurs whenever c_t is defined.

Figure 1



y_1
Length
$d_t - \varepsilon - x_0$
Length
$2q(\varepsilon + 2^{-t})$
P_3
P_2
D'
$f_e(t)$
y_0
(b_s, a_s)
x_0
P_0
P_1
x_1
Length
ε
d_t

Priority

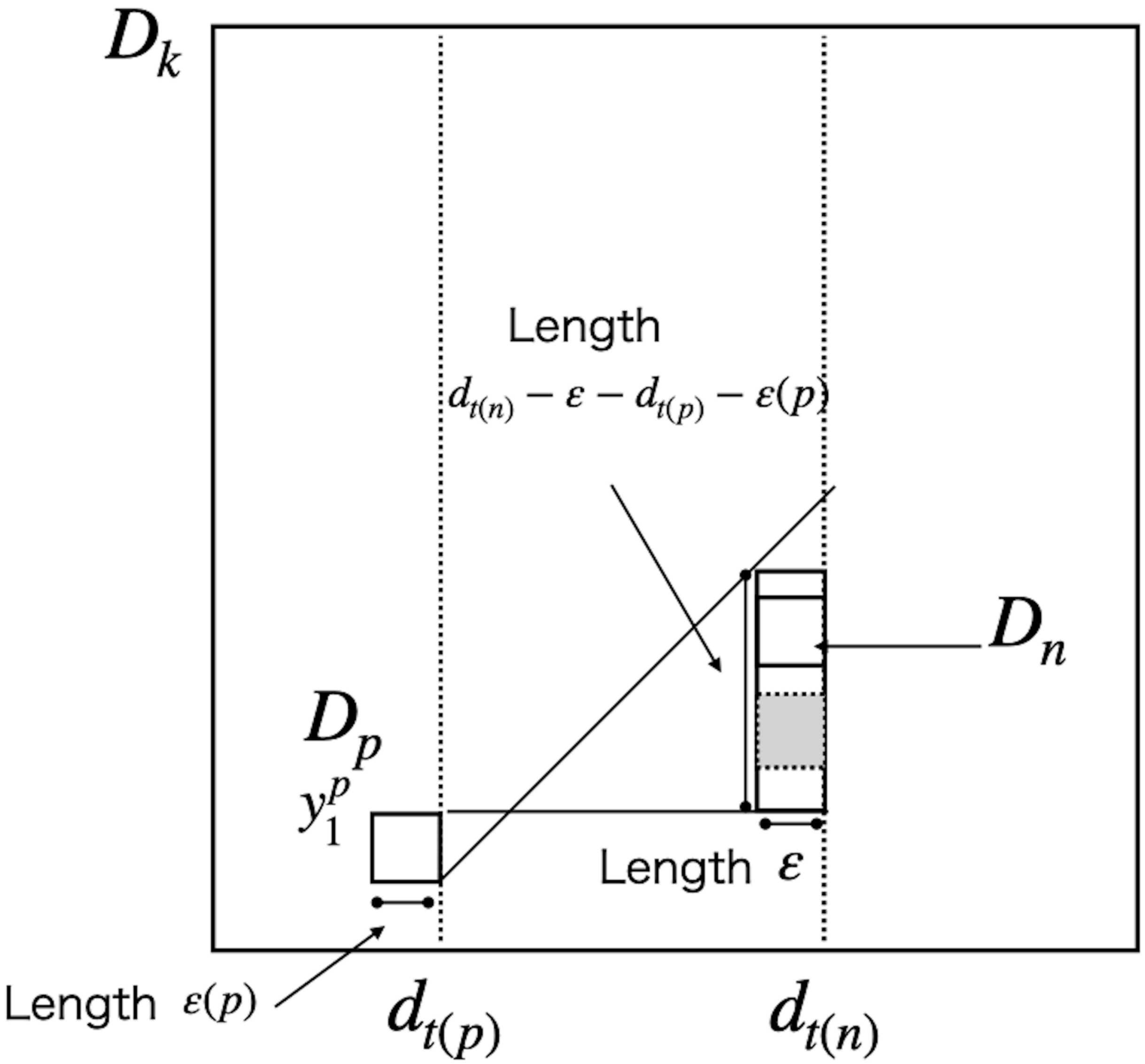
Enumerate all $f_e(t)$ for a sequence $(c_s)_s$ converging to α.

Assign priorities: $R_1 > R_2 > \cdots$.

Each requirement R_e has an associated forcing region D_e, satisfying $D_1 \supset D_2 \supset D_3 \supset \cdots$.

If $f_e(t)$ converges, then R_e may modify its forcing region D_e, destroying all requirements with lower priority.

Figure 2 ($k < n < p$)



D_k
Length
$d_{t(n)} - \varepsilon - d_{t(p)} - \varepsilon(p)$
D_n
D_p
y_1^p
Length ε
Length $\varepsilon(p)$
$d_{t(p)}$
$d_{t(n)}$

Finally

- By a similar method, one can also construct a counterexample where (I) holds but (III) fails.
- If the term 2^{-s} is replaced by any computable sequence converging to 0 (not necessarily monotone), the notion becomes robust.
- Solovay reducibility for left-c.e. reals can be characterized by Lipschitz functions; for c.a. reals, it requires “partial Lipschitz functions,” i.e., functions not extendable to the entire domain. This partiality manifests as the inability to interchange quantifiers.
- Be cautious when changing quantifiers.