SOLOVAY REDUCIBILITY FOR COMPUTABLY APPROXIMABLE REALS

KENSHI MIYABE

I will give a survey of recent developments in Solovay reducibility for computably approximable reals, focusing on connections to analytical concepts and their structural properties.

Solovay reducibility in the theory of algorithmic randomness [1, Chapter 9] compares two (mainly left-c.e.) reals in terms of approximability. We say that a real x is *computably approximable* or c.a. if there exists a computable sequence of rationals converging to x. Zheng and Rettinger [7] extended Solovay reducibility to c.a. reals as follows: Let α and β be c.a. reals. We say that α is Solovay reducible to β if, there exist computable sequences $(a_n)_n$ and $(b_n)_n$ of rationals converging to α, β respectively and $c \in \omega$ such that

$$|\alpha - a_n| \leq c(|\beta - b_n| + 2^{-n}) \text{ for all } n \in \omega.$$

Although Zheng and Rettinger called this notion S2a-reducibility, we simply refer to it as Solovay reducibility.

Solovay reducibility and Lipschitz coninuity

Solovay reducibility has a strong connection to Lipschitz continuity. In [2], we observed that Solovay reducibility for left-c.e. reals has a characterization via Lipschitz computable functions: For left-c.e. reals α and β, $\alpha \leq_S \beta$ if and only if there exists an increasing Lipschitz computable function $f : (-\infty, \beta) \to (-\infty, \alpha)$ such that $\lim_{x \to \beta - 0} f(x) = \alpha$. In [5], we extended it to for c.a. reals: For c.a. reals α and β, $\alpha \leq_S \beta$ if and only if there exists a semi-computable function interval (f, h) such that f, h are both Lipschitz functions and $f(\beta) = h(\beta) = \alpha$.

In the same paper, we characterize Solovay reducibility via the signed-digit representation: For c.a. reals α and β, $\alpha \leq_S \beta$ if and only if there is a partial function g such that $g(\beta) = \alpha$ and g is partially computable with respect to the signed-digit representation and with use bound $n + O(1)$. The notion with the binary representation has been called computably Lipschitz reducibility (cL-reducibility), which is different from Solovay reducibility. The partiality of g is important. If one replaces the partial function in the above statement with a total function, one obtains a different notion.

Variants of Solovay reducibility

Having established this fundamental characterization via Lipschitz functions, we now consider variations of Solovay reducibility corresponding to different levels of smoothness. This is actually the original motivation of [2]. In analysis, the following hierarchy is well known on a closed interval: C^1 implies Lipschitz continuity, which in turn implies Hölder continuity. With a correspondence, we introduced strong Solovay reducibility in [5] and quasi Solovay reducibility for left-c.e. reals in [2] and for c.a. reals in [5]. Then, quasi Solovay reducibility has a characterization via Hölder continuous functions. Strong Solovay reducibility has a characterization via zero derivative at the point, but not C^1, which is due to partiality of this notion.

Another motivation we introduced strong Solovay reducibility is that it is related with real closed fields. For any non-random left-c.e. real β, the set $\{\alpha \ : \ \alpha <_S \beta\}$ does not form a real closed field. In contrast, for any c.a. real β, the set $\{\alpha \ : \ \alpha \ll_S \beta\}$ does form a real closed field, where $\ll_S$ denotes strong Solovay reducibility. The details will appear in [4].

Quantifier Variations

An overlooked issue has been found concerning Solovay reducibility for c.a. reals. For left-c.e. reals α, β, $\alpha \leq_S \beta$ if and only if

$$\exists (a_n)_n \exists (b_n)_n \exists q \in \omega \forall n \in \omega [\alpha - a_n < q(\beta - b_n)],$$

where $(a_n)_n$ and $(b_n)_n$ varies increasing computable sequences of rationals converging to α and β, respectively. The first two quantifier parts can be replaced with $\forall (a_n)_n \exists (b_n)_n$ or $\forall (b_n)_n \exists (a_n)_n$. In this sense, Solovay reducibility for left-c.e. reals is robust.

This robustness does not apply for Solovay reducibility for c.a. reals. We can show that there are c.a. reals α and β such that the $\exists\exists$ statement holds but the $\forall\exists$ statment does not hold. This distinction highlights a fundamental difference between the left-c.e. case and the general c.a. case, suggesting that the notion of Solovay reducibility becomes more nuanced when extended beyond left-c.e. reals. The details will be presented in a forthcoming paper [3].

Variation randomness

A real x is called *weakly computable* if there is a computable sequence $(x_n)_n$ of rationals converging to x such that the variation $\sum_n |x_{n+1} - x_n|$ is finite. A real is weakly computable if and only if it is the difference between two left-c.e. reals. A weakly computable real is calld *variation non-random* if there is an approximation $(x_n)_n$ such that its variation is not ML-random, and is called *variarion random* otherwise, which are introduced by Miller [6].

Variation randomness is closely connected to Solovay reducibility. The key observation is the following: for a weakly computable real α, its variation β is always left-c.e. and we have $\alpha \leq_S \beta$. Conversely, for a weakly computable real α and for a left-c.e. real β, if $\alpha \leq_S \beta$, then there exists a computable approximation of α such that its variation is a multiple of β. With this fact, one can show that variation random reals are upward closed in Solovay degrees for weakly computable reals. We introduce a dual notion. We can show that there exists a weakly computable real β such that any left-c.e. real $\alpha \leq_S \beta$ must be computable.

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(K. Miyabe) Meiji University, Japan
Email address: research@kenshi.miyabe.name