Solovay reducibility for computably approximable reals

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Abstract

Solovay reducibility is a reducibility that caputures computability and randomness of reals.

Solovay reducibility for left-c.e. reals has been well studied, but it has not been well studied for c.a. reals and it behaves differently. The difference is due to the non-monotonicity and partiality.

It is roughly a reducibility induced by partially computable Lipschitz functions. Thus, it is also interesting to study it in the context of analysis.

Table of contents

Definition of Solovay reducibility

Computability of reals

$\alpha \in \mathbb{R}$ is **computable** if $\exists (a_n)_n$ comp. such that $|a_{n+1} - a_n| < 2^{-n}$ and $\lim_{n\to\infty} a_n = \alpha$.

α is **left-c.e.** if $\exists (a_n)_n$ comp. such that $(a_n)_n$ is increasing and $\lim_{n\to\infty} a_n = \alpha$.

α is **weakly computable** if $\exists (a_n)_n$ comp. such that its variation $\sum_n |a_{n+1} - a_n| < \infty$ and $\lim_{n\to\infty} a_n = \alpha$.

Proposition

α is weakly computable if and only if it is the difference of two left-c.e. reals.

Thus, weakly computable reals are sometimes called d.c.e. reals or d.l.c.e. reals.

α is **computably approximable** (c.a.) if $\exists (a_n)_n$ comp. such that $\lim_{n\to\infty} a_n = \alpha$.

Solovay reducibility for left-c.e. reals

α is **Solovay reducible** to β, denoted by $\alpha \leq_S \beta$, if $\exists f : \mathbb{Q} \to \mathbb{Q}$ partial comp. func. and $\exists c \in \omega$ such that

$$q \in \mathbb{Q}, q < \beta \Rightarrow f(q) \downarrow < \alpha, \ \alpha - f(q) < c(\beta - q)$$

(Solovay 1975)

If given a good approximation q of β from below, we can compute a good approximation of α from below.

Some characterizations

Let α, β be left-c.e. reals. Then, the following are equivalent:

- $\alpha \leq_S \beta$
- $\exists (a_n)_n \uparrow \alpha, \exists (b_n)_n \uparrow \beta$ comp. and $\exists c \in \omega$ such that

$$\alpha - a_n < c(\beta - b_n), \quad \forall n \in \omega.$$

- $\exists (a_n)_n \uparrow \alpha \exists (b_n)_n \uparrow \beta$ comp. and $\exists c \in \omega$ such that

$$a_{n+1} - a_n < c(b_{n+1} - b_n), \quad \forall n \in \omega.$$

Algebraic characterization

Theorem (Downey-Hirschfeldt-Nies 2002)
Let α, β be left-c.e. reals. Then, $\alpha \leq_S \beta$ if and only if $\exists \gamma$ left-c.e. real, $\exists q \in \omega$ such that

$$\alpha + \gamma = q\beta$$

Basic properties

- If $\alpha \leq_S \beta$, then $\alpha \leq_T \beta$ where $\leq_T$ denotes Turing reducibility.
- If $\alpha \leq_S \beta$, then $\alpha \leq_K \beta$ where K denotes prefix-free Kolmogorov complexity.

Theorem (Kučera-Slaman, Solovay, Calude-Hertling-Khoussainov-Wang, Downey-Hirschfeldt-Miller-Nies)

Among left-c.e. reals, the top Solovay degrees contain exactly ML-random reals.

Solovay reducibility for left-c.e. reals is well-behaved, but it is not for outside.

Solovay reducibility for c.a. reals

Let α, β be comp. approx. reals.

α is **Solovay reducible** to β, denoted by $\alpha \leq_S \beta$, if $\exists (a_n)_n \to \alpha, \exists (b_n)_n \to \beta$ comp. and $\exists c \in \omega$ such that

$$|\alpha - a_n| < c(|\beta - b_n| + 2^{-n}), \quad \forall n \in \omega.$$

Zheng and Rettinger (2004) introduced this notion with the name of $S2a$-reducibility.

This definition conincides with the original definition for left-c.e. reals.

I believe this is the correct definition and thus calle it just Solovay reducibility.

ML-randomness in Solovay reducibility

Theorem (Rettinger and Zheng 2005)

Let α be a weakly comp. real. If α is ML-random, then α is left-c.e. or right-c.e.

Corollary

Among weakly computable reals, the top Solovay degrees contain exactly ML-random reals.

Characterization via Lipschitz functions

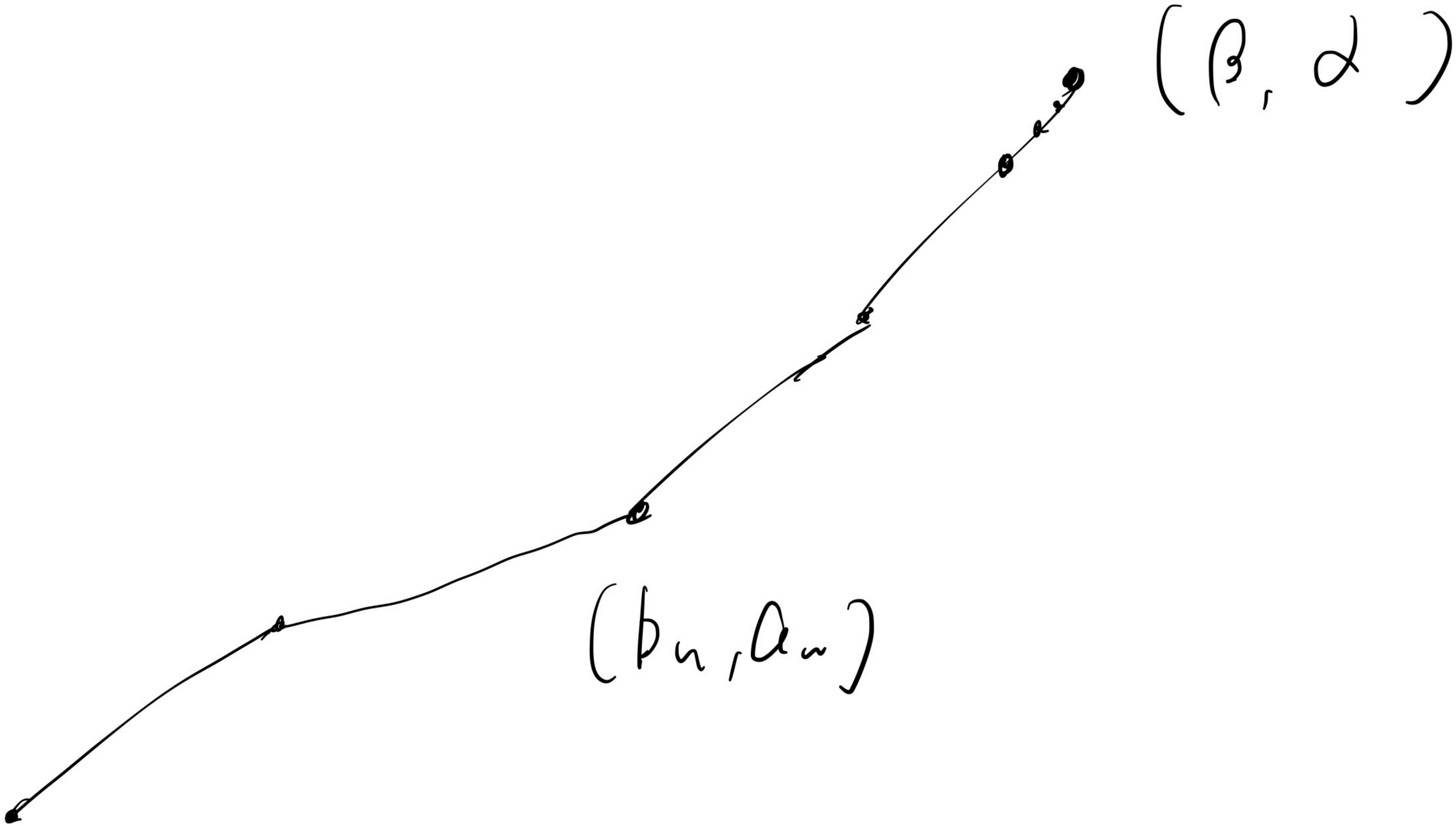
Characterization via Lipschitz functions

Proposition (Kumabe, M., Mizusawa, and Suzuki 2020)
Let α, β be left-c.e. reals. Then $\alpha \leq_S \beta$ if and only if there exists a computable increasing Lipschitz function $f :\subseteq [0, \beta) \to [0, \alpha)$ such that

$$\lim_{x \to \beta^-} f(x) = \alpha.$$

Remark

This part is due to Dr. Mizusawa and Prof. Suzuki.

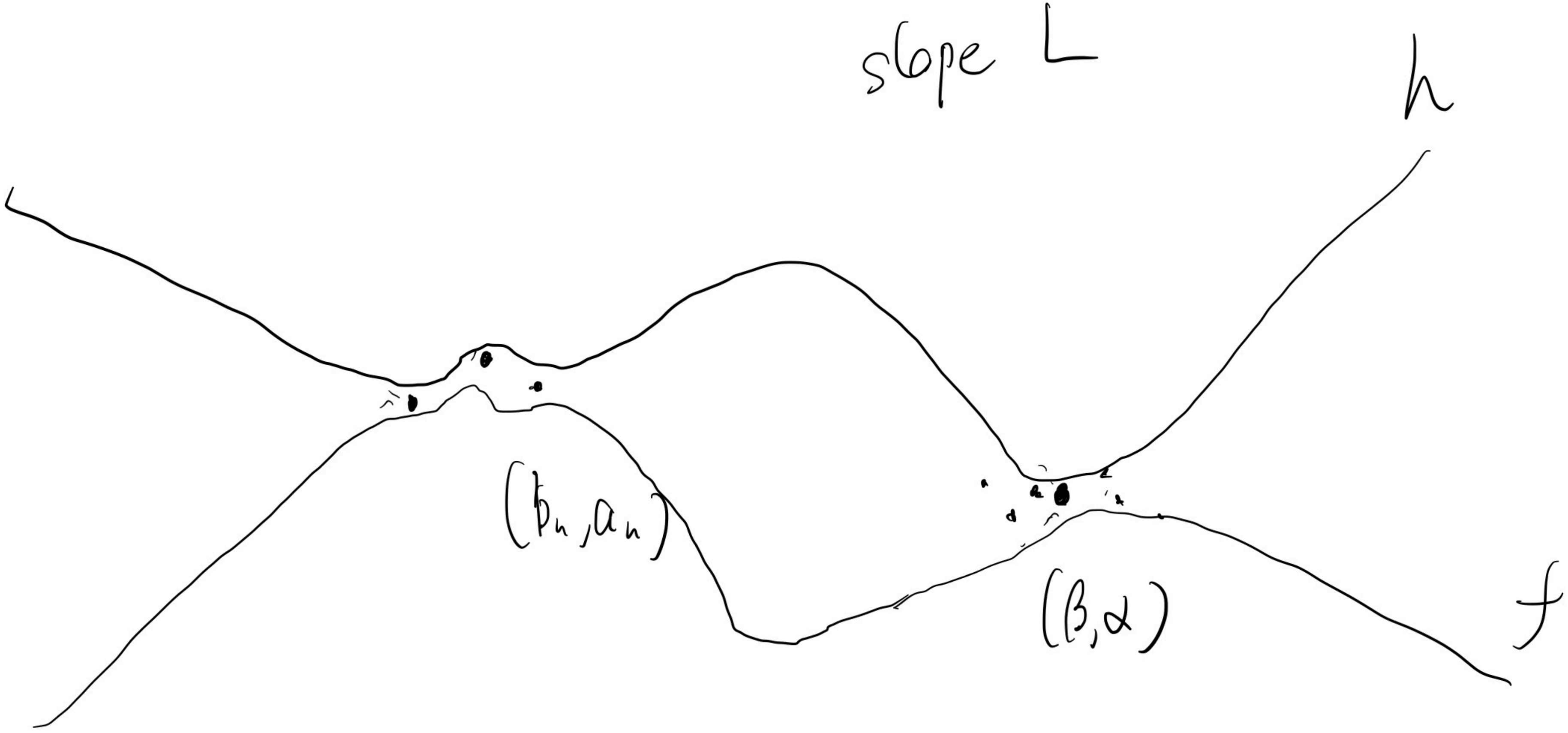

(β, α)
(b_n, a_n)

Characterization via Lipschitz functions

Definition (Kumabe, M., and Suzuki; Lipschitz-paper)
A **function interval** is a pair of functions (f, h) such that $f(x) \leq h(x)$ for all $x \in \mathbb{R}$. A function interval (f, h) is **semi-computable** if f is lower semi-computable and h is upper semi-computable.

Theorem (Lipschitz-paper)
Let α, β be c.a. reals. Then $\alpha \leq_S \beta$ if and only if there exists a semi-computable function interval (f, h) such that

- f and h are both Lipschitz continuous functions,
- $f(\beta) = h(\beta) = \alpha$.


slope L
h
(b_n, a_n)
(β, α)
f

Characterization via Lipschitz functions

Remark

We can not replace it a computable Lipschitz function.
(A proof is by the priority argument.)
This fact reflects the non-monotonicity and the partiality of Solovay reducibility.

Solovay reducibility is roughly a reducibility induced by **partially computable Lipschitz functions**.

cL-reducibility

Definition (computable Lipschitz reducibility)
Let $\alpha, \beta \in 2^\omega$. α is **cL-reducible** to β, denoted by $\alpha \leq_{cL} \beta$, if there exists a Turing functional Φ such that $\alpha = \Phi(\beta)$ and $\mathrm{use}(\Phi, \beta, n) \leq n + O(1)$.

Theorem (Kumabe, M., and Suzuki; Lipschitz-paper)
Let α, β be c.a. reals. Then $\alpha \leq_S \beta$ if and only if there exists a partial computable functional g with respect to signed-digit representation such that $\alpha = \Phi(\beta)$ and $\mathrm{use}(g, \beta, n) \leq n + O(1)$.

Cauchy-type characterization

Proposition (Kumabe, M., and Suzuki; Lipschitz-paper)
Let α, β be c.a. reals. Then $\alpha \le_S \beta$ if and only if $\exists (a_n)_n \to \alpha, \exists (b_n) \to \beta$ comp. and $\exists c \in \omega$ such that

$$(\forall k, n \in \omega)[k < n \implies |a_n - a_k| < c(|b_n - b_k| + 2^{-k})].$$

Quantifier variations

Robustness

Observation

Let α, β be left-c.e. reals. The following are equivalent:

- $\exists (a_n)_n \exists (b_n)_n P$
- $\forall (a_n)_n \exists (b_n)_n P$
- $\forall (b_n)_n \exists (a_n)_n P$

where $P = \exists c \in \omega \forall n \in \omega [\alpha - a_n < c(\beta - b_n)]$.
Here, $(a_n)_n$ and $(b_n)_n$ are comp. approx. from below of α and β, respectively.

In this sense, Solovay reducibility for left-c.e. reals is robust.

Non-robustness for c.a. reals

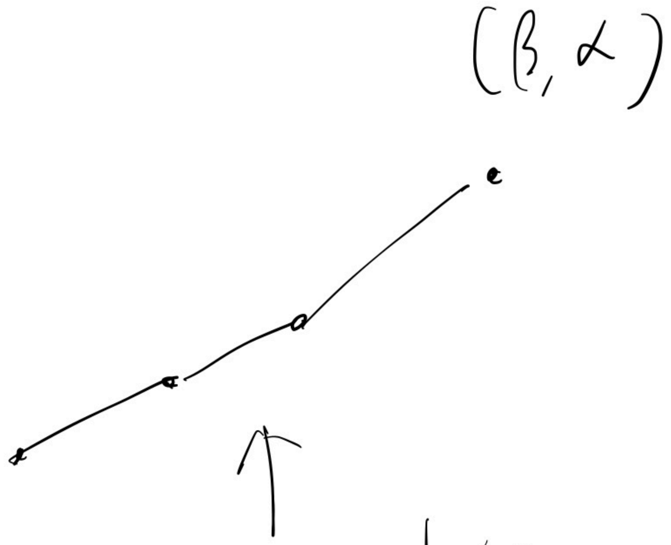
- (I) $\exists (a_n)_n \exists (b_n)_n P$
- (II) $\forall (b_n)_n \exists (a_n)_n P$
- (III) $\forall (a_n)_n \exists (b_n)_n P$

where $P = \exists c \in \omega \forall n \in \omega [|\alpha - a_n| < c(|\beta - b_n| + 2^{-n})]$.
Here, $(a_n)_n$ and $(b_n)_n$ are comp. approx. of α and β, respectively.

Theorem
(I) does not imply (II) or (III).

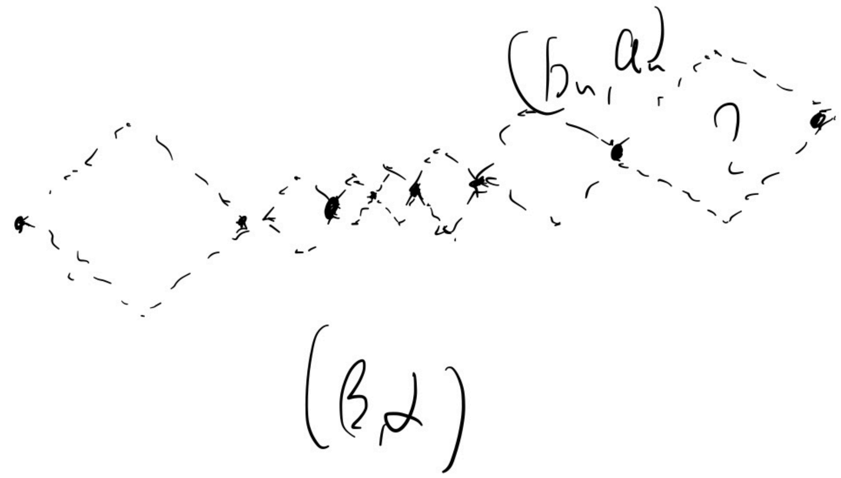
The proof is by the priority argument. In fact, we further impose α, β to be left-c.e.

left-c.e.



can interpolate.

c.a.



somewhere in the boxes

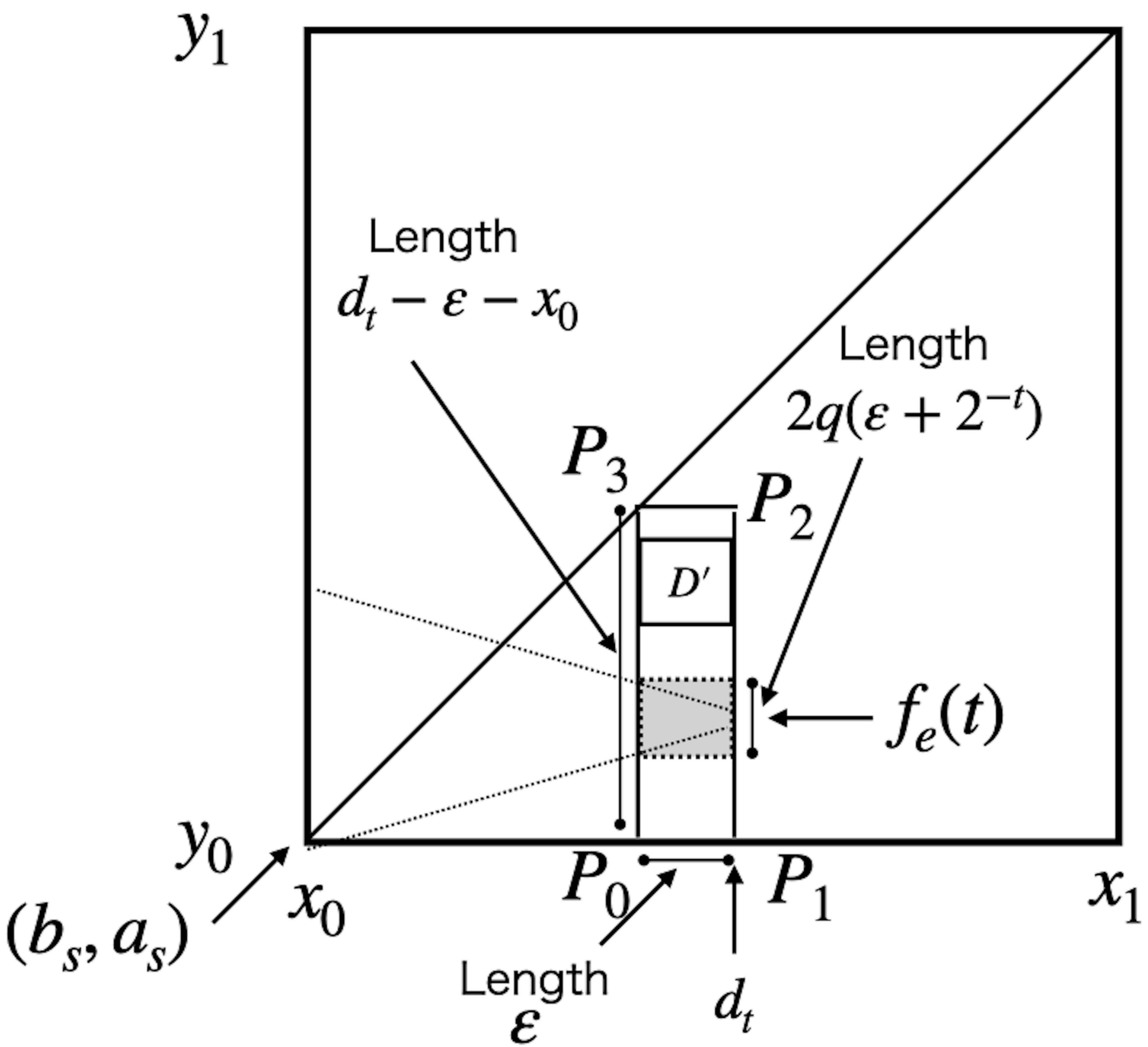
Proof idea

Negation of (II):

$$\exists (d_s)_s \forall (c_s)_s \forall q \in \omega \exists t \in \omega [|\alpha - c_t| \geq q(|\beta - d_t| + 2^{-t})]$$

- We need to determine d_t first.
- c_t will be determined afterwards.
- Then we force β to be close to d_t and let α be away from c_t.

In order for $\alpha \leq_S \beta$, and change α, we must also change β. Therefore, d_t and b_s must be kept apart.


y_1
Length
$d_t - \varepsilon - x_0$
Length
$2q(\varepsilon + 2^{-t})$
P_3
P_2
D'
$f_e(t)$
y_0
(b_s, a_s)
x_0
P_0
P_1
x_1
Length
ε
d_t

Strong Solovay reducibility

Strong Solovay reducibility

Definition (Kumabe, M., and Suzuki; RCF-paper)

Let α, β be c.a. reals. α is **strongly Solovay reducible** to β, denoted by $\alpha \ll_S \beta$, if $\exists (a_n)_n \to \alpha, \exists (b_n)_n \to \beta$ comp. such that

$$\frac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}} \to 0 \text{ as } n \to \infty.$$

Remark

When giving a definition, one can choose which quantifiers are applied. The paper by Imai, Kumabe, M., Mizusawa, and Suzuki (2022) defined strong Solovay reducibility for left-c.e. reals with $\forall\forall$ quantifiers.

terminology

Why did I choose this name?

Proposition (RCF-paper)
Let α, β be left-c.e. reals. Then,

$$\alpha \ll_K \beta \implies \alpha \ll_S \beta \implies \alpha \leq_S \beta \implies \alpha \leq_K \beta.$$

Here, $\alpha \ll_K \beta$ if $K(\beta \upharpoonright n) - K(\alpha \upharpoonright n) \to \infty$ as $n \to \infty$, which is called strong K-reducibility.

Proposition (RCF-paper)
The first implication does not hold in general for weakly comp. reals.

Real closed field

Consider the set

$$\{\alpha : \alpha <_S \beta\}$$

If $\beta = \Omega$, this set forms a real closed field (by a result by Miller). If β is a non-ML-random left-c.e. real, then this does not form a real closed field.

Consider the set

$$\{\alpha : \alpha \leq_S \beta\}, \quad \{\alpha : \alpha \ll_S \beta\}$$

For each weakly comp. real β, this set forms a real closed field. Furthermore, $\{\alpha : \alpha <_S \Omega\} = \{\alpha : \alpha \ll_S \Omega\}$.

Characterization via derivative 0

Theorem (RCF-paper)
Let α, β be c.a. reals. Then $\alpha \ll_S \beta$ if and only if $\exists (a_n)_n \exists (b_n)_n$ comp. approx. of α and β, respectively, and a continuous function g such that

- the derivative $g'(\beta) = 0$,
- $|g(b_n) - a_n| \leq 2^{-n}$ for all $n \in \omega$.

g can be chosen to be differentiable.

Remark
g need not be computable.

quasi Solovay reducibility

Definition (quasi Solovay reducibility; RCF-paper)

Let α, β be c.a. reals. Then α is **quasi Solovay reducible** to β, denoted by $\alpha \le_{qS} \beta$, if $\exists (a_n)_n \exists (b_n)_n$ comp. and $\exists s, q \in \mathbb{Q}_{>0}$ such that

$$|\alpha - a_n| \le q(|\beta - b_n|^s + 2^{-n}), \quad \forall n \in \omega.$$

Theorem (RCF-paper)

Let α, β be c.a. reals. Then $\alpha \le_{qS} \beta$ if and only if there exists a semi-computable function interval (f, h) such that

- f and h are both s-Hölder continuous functions for some $s \in (0, 1]$,
- $f(\beta) = h(\beta) = \alpha$.

Question

Question

Can one impose g to be C^1?

In classical analysis, on a compact interval,

$$C^1 \implies \text{Lipschitz continuous} \implies \text{Hölder continuous}\ ,$$

but the derivative 0 does not imply Lipschitz continuity.

Again, by the partiality of Solovay reducibility, the hierarchy does not fit exactly.

Variations in Solovay reducibility

Variation randomness

Let $(a_n)_n$ be a comp. approx. of a weakly comp. real α. The total variation of $(a_n)_n$ is defined by

$$V_0((a_n)_n) = |a_0| + \sum_{n=0}^{\infty} |a_{n+1} - a_n|.$$

Definition (Miller 2017)

A weakly comp. real α is called a **variation random** if each total variation is ML-random.

Variation randomness

Theorem (Miller 2017)
There exists a weakly comp. real α such that α is not ML-random but α is variation random.

Theorem (Miller 2017)
A weakly comp. real α is not variation random if and only if it is the difference of two non-ML-random left-c.e. reals.

Algebraic characterization

Theorem (in preparation)
Let α be a weakly comp. real and β be a left-c.e. real. Then, the following are equivalent:

- $\alpha \leq_S \beta$,
- $\exists (a_n)_n$ comp. approx. of α, $\exists q \in \omega$ comp. such that $V_0((a_n)_n) = q\beta$,
- $\exists \gamma, \delta$ left-c.e. reals, $q \in \omega$ such that $\gamma + \delta = q\beta$ and $\gamma - \delta = \alpha$.

Algebraic characterization

Theorem (Downey-Hirschfeldt-Nies 2002)
Let α, β be left-c.e. reals. Then, $\alpha \leq_S \beta$ if and only if $\exists \gamma$ left-c.e. real, $\exists q \in \omega$ such that $\alpha + \gamma = q\beta$.

Dual notion

Question

How about the case that α is left-c.e. and β is weakly comp.?

This is an ongoing work. A result we have so far is the following:

Theorem

There exists a weakly comp. real β that is **quasi-minimal** relative to left-c.e. reals, that is, if $\alpha \leq_S \beta$ and α is left-c.e., then α is computable.

Thank you for your listening!