Variation of weakly computable reals in Solovay reducibility

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1-page Summary

Solovay reducibility is a reducibility that caputures computability and randomness of reals.

Solovay reducibililty for **left-c.e. reals** has a good algebraic characterization.

Zheng and Rettinger (2004) introduced Solovay reducibility for **computably approximable (c.a.) reals**.

The main contribution is an **algebraic characterization** of $\alpha \leq_{SZR} \beta$ where α is weakly computable and β is left-c.e.

This is a natural extension of a characterization of variation randomness by Miller (2017).

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Overview

Random sequences

$$\alpha = 0.000100100101101010001011101010$$

$$\beta = 0.000000010000010010000010110010$$

$$\gamma = 0.000000000000000000000000000000$$

- Which sequence is random?
- Which sequence is more random?

If good approximation of α can be computed from an approximation of β, then α is not more random than β.

-> Solovay reducibility

Solovay reducibility

Solovay reducibility captures computability and randomness of reals.

- If $\alpha \leq_S \beta$ and β is computable, then α is computable.
- If $\alpha \leq_S \beta$ and α is ML-random, then β is ML-random.

Roughly speaking, having the same Solovay degree means haveing the same information.

Left-c.e. ML-random

Kučera-Slaman Theorem (2001)
All left-c.e. ML-random reals have the Solovay complete degree.

Among left-c.e. reals, only one type of ML-random reals exist in terms of Solovay reducibility.

Algebraic characterization

Theorem (Downey-Hirschfeldt-Nies 2002)
Let α, β be left-c.e. reals. Then, $\alpha \leq_S \beta$ if and only if $\exists \gamma$ left-c.e. real, $\exists q \in \omega$ such that

$$\alpha + \gamma = q\beta$$

One interpretation is that Solovay reduciblity captures the convergence speed.
If $\alpha \leq_S \beta$, then α converges converges no faster than β.

Question

Do we have a similar algebraic characterization for weakly computable reals?

More specifically,

- How related are the convergence speeds of weakly computable reals and its Solovay degree?

-> We give a partial affirmative answer and extend the result of variation randomness by Miller (2017).

Solovay reducibility for left-c.e. reals

Computability of reals

$\alpha \in \mathbb{R}$ is **computable** if $\exists (a_n)_n$ comp. such that $|a_{n+1} - a_n| < 2^{-n}$ and $\lim_{n\to\infty} a_n = \alpha$.

α is **left-c.e.** if $\exists (a_n)_n$ comp. such that $(a_n)_n$ is increasing and $\lim_{n\to\infty} a_n = \alpha$.

α is **weakly computable** if $\exists (a_n)_n$ comp. such that its variation $\sum_n |a_{n+1} - a_n| < \infty$ and $\lim_{n\to\infty} a_n = \alpha$.

Proposition

α is weakly computable if and only if it is the difference of two left-c.e. reals.

Thus, weakly computable reals are sometimes called d.c.e. reals or d.l.c.e. reals.

α is **computably approximable** (c.a.) if $\exists (a_n)_n$ comp. such that $\lim_{n\to\infty} a_n = \alpha$.

Solovay reducibility for left-c.e. reals

α is **Solovay reducible** to β, denoted by $\alpha \leq_S \beta$, if $\exists f : \mathbb{Q} \to \mathbb{Q}$ partial comp. func. and $\exists c \in \omega$ such that

$$q \in \mathbb{Q}, q < \beta \Rightarrow f(q) \downarrow < \alpha, \alpha - f(q) < c(\beta - q)$$

(Solovay 1975)

If given a good approximation q of β from below, we can compute a good approximation of α from below.

Some characterizations

Let α, β be left-c.e. reals. Then, the following are equivalent:

- $\alpha \leq_S \beta$
- $\exists (a_n)_n \uparrow \alpha, \exists (b_n)_n \uparrow \beta$ comp. and $\exists c \in \omega$ such that

$$\alpha - a_n < c(\beta - b_n), \quad \forall n \in \omega.$$

- $\exists (a_n)_n \uparrow \alpha \exists (b_n)_n \uparrow \beta$ comp. and $\exists c \in \omega$ such that

$$a_{n+1} - a_n < c(b_{n+1} - b_n), \quad \forall n \in \omega.$$

Algebraic characterization

Theorem (Downey-Hirschfeldt-Nies 2002)
Let α, β be left-c.e. reals. Then, $\alpha \leq_S \beta$ if and only if $\exists \gamma$ left-c.e. real, $\exists q \in \omega$ such that

$$\alpha + \gamma = q\beta$$

Basic properties

- If $\alpha \leq_S \beta$, then $\alpha \leq_T \beta$ where $\leq_T$ denotes Turing reducibility.
- If $\alpha \leq_S \beta$, then $\alpha \leq_K \beta$ where K denotes prefix-free Kolmogorov complexity.

Theorem (Kučera-Slaman, Solovay, Calude-Hertling-Khoussainov-Wang, Downey-Hirschfeldt-Miller-Nies)

Among left-c.e. reals, the top Solovay degrees contain exactly ML-random reals.

Solovay reducibility for left-c.e. reals is well-behaved, but it is not for outside.

Solovay reducibility for c.a. reals

Solovay reducibility for c.a. reals

Let α, β be comp. approx. reals.

α is **Solovay reducible** to β, denoted by $\alpha \leq_S \beta$, if $\exists (a_n)_n \to \alpha, \exists (b_n)_n \to \beta$ comp. and $\exists c \in \omega$ such that

$$|\alpha - a_n| < c(|\beta - b_n| + 2^{-n}), \quad \forall n \in \omega.$$

Zheng and Rettinger (2004) introduced this notion with the name of $S2a$-reducibility.

This definition conincides with the original definition for left-c.e. reals.

I believe this is the correct definition and thus calle it just Solovay reducibility.

ML-randomness in Solovay reducibility

Theorem (Rettinger and Zheng 2005)
Let α be a weakly comp. real. If α is ML-random, then α is left-c.e. or right-c.e.

Corollary
Among weakly computable reals, the top Solovay degrees contain exactly ML-random reals.

Variation randomness

Variation randomness

The original motivation is to capture variation randomness by Solovay reducibility.

Let $(a_n)_n$ be a comp. approx. of a weakly comp. real α. The total variation of $(a_n)_n$ is defined by

$$V_0((a_n)_n) = |a_0| + \sum_{n=0}^{\infty} |a_{n+1} - a_n|.$$

Definition (Miller 2017)

A weakly comp. real α is called a **variation random** if each total variation is ML-random.

Variation randomness

Theorem (Miller 2017)
There exists a weakly comp. real α such that α is not ML-random but α is variation random.

Theorem (Miller 2017)
A weakly comp. real α is not variation random if and only if it is the difference of two non-ML-random left-c.e. reals.

Algebraic characterization

Here is our main result.

Theorem

Let α be a weakly comp. real and β be a left-c.e. real. Then, the following are equivalent:

1. $\alpha \leq_S \beta$,
2. $\exists (a_n)_n$ comp. approx. of α, $\exists q \in \omega$ comp. such that $V_0((a_n)_n) = q\beta$,
3. $\exists \gamma, \delta$ left-c.e. reals, $q \in \omega$ such that $\gamma + \delta = q\beta$ and $\gamma - \delta = \alpha$.

Some thought

$2 \iff 3$ is a generalization of Miller's characterization of variation randomness when β is taken to be not a ML-random real.

Being both non-ML-random and variation random means being a quasi-maximal element relative to left-c.e. reals in Solovay degrees.

Algebraic characterization

Theorem (Downey-Hirschfeldt-Nies 2002)
Let α, β be left-c.e. reals. Then, $\alpha \leq_S \beta$ if and only if $\exists \gamma$ left-c.e. real, $\exists q \in \omega$ such that $\alpha + \gamma = q\beta$.

Proof sketch

$1 \Rightarrow 2$:

α is weakly comp., β is left-c.e., $\alpha \leq_S \beta$.

There exist $(a_n)_n \to \alpha$, $(b_n)_n \uparrow \beta$ comp., $q \in \omega$ such that

$$|a_{n+1} - a_n| < q(b_{n+1} - b_n)$$

for all $n \in \omega$. By adding some redundant move to $(a_n)_n$, we have the desired result.

Proof sketch

$2 \Rightarrow 3$:

In the sequence $(a_n)_n$, divide it into the parts where it moves upward and where it moves downward, and denote their respective sums by γ and δ.

$3 \Rightarrow 1$:

If one has good approximation of β, then one can compute good approximations of γ and δ, and hence of α.

Question

For a weakly comp. real α, being a left-c.e. real above α in the sense of Solovay reducibility is equivalent to being the total variation of some approximation sequence of α.

Question:
What does it mean that being a left-c.e. real below α in the sense of Solovay reducibility?

Some Idea

- Left-c.e. reals have **complete information** but weakly comp. reals only have **partial information**; actually, weakly comp. reals can be quasi-minimal relative to left-c.e. reals in Solovay degrees.
- For left-c.e. reals, being difficult to approximate is roughly the same as having much information.
- For weakly comp., these two notions are different; for example, variation randomness.

Thank you for your listening!