Randomness with respect to c.e. semimeasures

Kenshi Miyabe (Meiji University)
宮部賢志(明治大学)

CCA2025 @ RIMS
24-26 Sep 2025

This research was initiated by a question posed by Laurent Bienvenu while he was on sabbatical in Japan from July 2023 to July 2024.

1-page Summary

Question: What is randomness with respect to c.e. semimeasures?

Answer: We have four different notions of randomness; three of them are based on complexity and the other is based on tests. We prove the implications and separations among them.

Each of notions has been studied in the context of partial randomness, but different notions turn out to be equivalent.

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Martin-Löf Randomness

μ: uniform measure on 2^ω.

$U \subseteq 2^\omega$ c.e. open set = $\bigcup_i [\sigma_i]$ where $\sigma_i \in 2^{<\omega}$.

ML-test $(U_n)_n$: uniformly c.e. open sets with $\mu(U_n) \leq 2^{-n}$.

$X \in 2^\omega$ is **ML-random** if $X \notin \bigcap_n U_n$ for any ML-test $(U_n)_n$.

Levin-Schnorr theorem

Levin-Schnorr theorem: the following are equivalent.

- X is ML-random.
- $K(X \upharpoonright n) \geq n - O(1)$.
- $KA(X \upharpoonright n) \geq n - O(1)$.

Here, K is prefix-free Kolmogorov complexity and KA is a priori complexity.

Randomness w.r.t. computable measures

A measure μ on 2^ω is determined by the function $\sigma \mapsto \mu([\sigma])$.

μ is **computable** if this function is computable.

As a natural extention of Levin-Schnorr's theorem, the following are equivalent.

- X is ML-random w.r.t. μ.
- $K(X \upharpoonright n) \geq -\log \mu(X \upharpoonright n) - O(1)$.
- $KA(X \upharpoonright n) \geq -\log \mu(X \upharpoonright n) - O(1)$.

Let $f(\sigma) = -\log \mu([\sigma])$.

Randomness w.r.t. c.e. semimeasures

c.e. semimeasure

semimeasure: $\mu : 2^{<\omega} \to [0,1]$ such that

$$\mu(\varepsilon) \leq 1, \quad \mu(\sigma) \geq \mu(\sigma 0) + \mu(\sigma 1) \text{ for all } \sigma.$$

A semimeasure is **c.e.** if the function is lower semicomputable.

Important notion because

- one can enumerate all c.e. semimeasures but not all computable measures,
- there is a correspondence with a c.e. martingale, which characterizes Martin-Löf randomness,
- it is used in the definition of a priori complexity KA.

Randomness w.r.t. c.e. semimeasures

Question: What is randomness with respect to a c.e. semimeasure?

Before answering this question, let us ask why we need such a notion.

Answer 1: Any computable function defined a.e. induces randomness w.r.t. a computable measure; randomness preservation and no-randomness-from nothing. But a partial computable function induces a c.e. semimeasure. (Bienvenu, Hölzl, Porter, and Shafer 2017)

Answer 2: It gives another proof of Kučera-Gács theorem. (Barmpalias and Shen 2023)

Randomness w.r.t. c.e. semimeasures

$f : 2^{<\omega} \to [0,\infty]$, usually $f = -\log \mu$ for a c.e. semimeasure μ

(I) KA-f-complex: $KA(X \upharpoonright n) > f(X \upharpoonright n) - O(1)$.

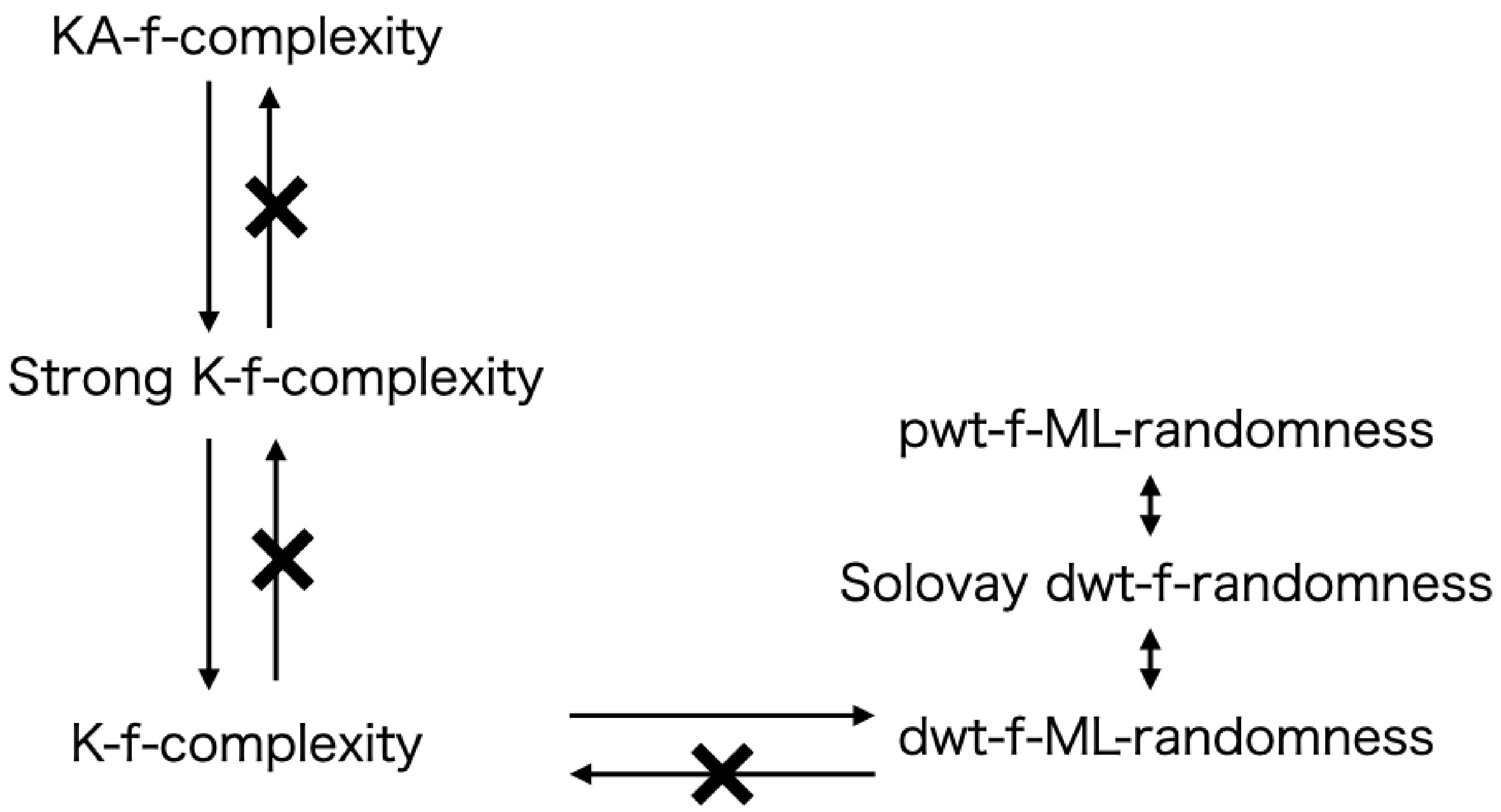
(II) strongly K-f-complex: $K(X \upharpoonright n) - f(X \upharpoonright n) \to \infty$ as $n \to \infty$.

(III) K-f-complex: $K(X \upharpoonright n) > f(X \upharpoonright n) - O(1)$.

(IV) f-ML-test: uniformly c.e. sets $S_n \subseteq 2^{<\omega}$ with $\sum_{\sigma \in S_n} 2^{-f(\sigma)} \le 2^{-n}$ for all n.
f-ML-random if $X \notin \bigcap_n [S_n]$ for any f-ML-test $(S_n)_n$.

(IV) studied by Bienvenu et al. (2017),
(I) with some conditions used by Barmpalias and Shen (2023).

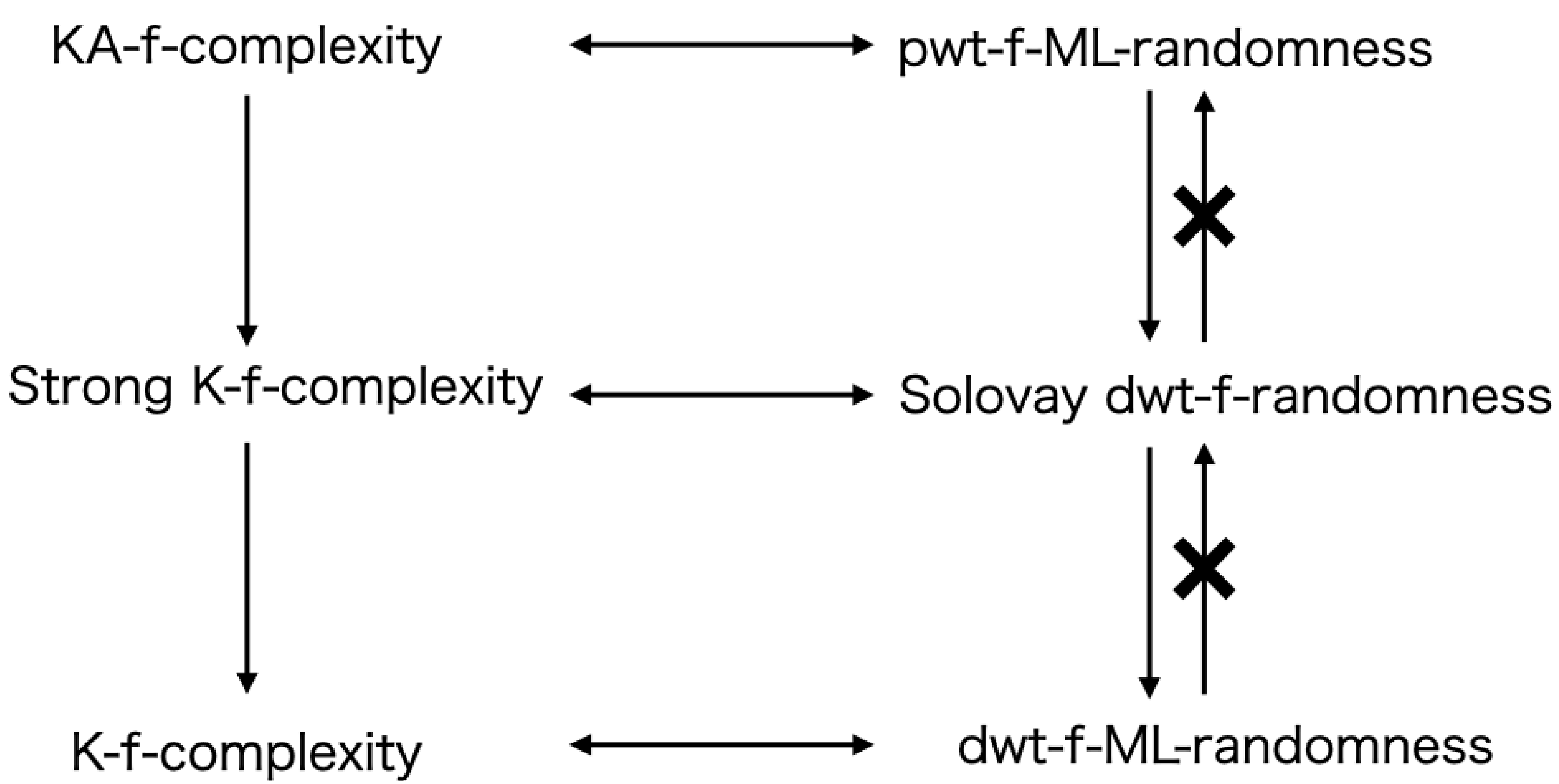
When f is upper semicomputable and concave



Comparison

	rand. w.r.t. c.e. semimeas.	partial randomness
comp. of f	upper semicomp.	computable
shape of f	concave	convex

When f is computable by Hudelson (2013)



Randomness preservation

Theorem (M.)
Let f be the function induced by a c.e. semimeasure. Randomness preservation holds for strong K-f-complexity but not for KA-f-complexity in general.

It holds for (II) but not for (I).

No-randomness-from-nothing

Barmaplias and Shen (2023) proved no-randomness-from-nothing for (I) with use bound $2n + O(1)$.

Theorem (M.)
No-randomness-from-nothing does not hold for strong K-f-complexity even with use bound $n + O(1)$ in general.

It holds for (I) with use bound but not for (II) even with use bound.

Proof

KA and K

According to Andreev and Kumon (2016), the follwoing result is given by Lempp, Miller, Ng, and Turetsky (2010) in an unpublished manuscript.

Theorem

Let $X \in 2^\omega$. Then

$$K(X \restriction n) - KA(X \restriction n) \to \infty \text{ as } n \to \infty.$$

As a corollary, we have (I) $\Rightarrow$ (II).

We give a proof in the paper. The proof idea is from Higuchi, Hudelson, Simpson, Yokoyama (2014).

Proof

Let $X \in 2^\omega$.

By Kučera-Gács theorem, there exists $Y \in 2^\omega$ such that $X \leq_T Y \in \mathrm{MLR}$.

Let Φ be a Turing functional such that $X = \Phi(Y)$.

For each $\sigma \in 2^{<\omega}$, let

$$V_\sigma = \{\overline{Y} \ : \ \sigma \preceq \Phi^{\overline{Y}}\}.$$

We define a c.e. semimeasure ν by $\nu(\sigma) = \mu(V_\sigma)$.

Let ξ be an optimal c.e. semimeasure such that $KA = -\log \xi$. Then, there exists $c_0 \in \omega$ such that $\mu(V_\sigma) = \nu(\sigma) \leq c_0 \xi(\sigma)$ for all $\sigma \in 2^{<\omega}$.

Proof

Suppose that there exists $c_1 \in \omega$ such that

$$K(X \upharpoonright n) < KA(X \upharpoonright n) + c_1$$

for infinitely many n. Then, $\mu(V_{X \upharpoonright n}) < c_0 2^{c_1 - K(X \upharpoonright n)}$ for infinitely many n.

Let U be the universal prefix-free machine to define K. Consider

$$W = \bigcup_{\tau \in \mathrm{dom}(U)} \widehat{V_{U(\tau)}},$$

where $\widehat{V_{U(\tau)}}$ is $V_{U(\tau)}$ enumerated as long as $\mu(V_{U(\tau)}) < c_0 2^{c_1 - |\tau|}$.

Proof

The weight of W is bounded by

$$\sum_{\tau \in \mathrm{dom}(U)} \mu(\widehat{V_{U(\tau)}}) \leq \sum_{\tau \in \mathrm{dom}(U)} c_0 2^{c_1 - |\tau|} < \infty.$$

By the assumption, there are infinitely many $\sigma \in X$ such that

$$Y \in V_\sigma, \mu(V_\sigma) < c_0 2^{c_1 - K(\sigma)}.$$

Thus, there are infinitely many $\tau \in \mathrm{dom}(U)$ such that

$$Y \in \widehat{V_{U(\tau)}} = V_{U(\tau)}.$$

Proof

Then, Y is not ML-random, a contradiction.

Comment:
The proof is based on Theorem 4.5 in Higuchi, Hudelson, Simpson, Yokoyama (2014). The theorem is about KA-f-complexity. My contribution is to show that we can remove the function f from the theorem and its proof.

Thank you for your listening!