ON THE LOWER DENSITY OF SOLOVAY–ZHENG–RETTINGER DEGREES

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Abstract. In the theory of algorithmic randomness, it is well known that the Solovay degrees of left-c.e. reals are dense. In this paper, we establish a corresponding lower density result for degrees of the modified version of Solovay reducibility introduced by Zheng and Rettinger. It is known that the modified reducibility behaves better for computably approximable reals than the orignal reducibility. We call the modified one Solovay–Zheng–Rettinger reducibility (abbreviated as Solovay–ZR reducibility). Our proof employs a completely different strategy from the known argument. Furthermore, we demonstrate the existence of a quasi-minimal Solovay–ZR degree: a weakly computable real such that every left-c.e. real Solovay–ZR-reducible to it is necessarily computable. Finally, we point out that this notion can be regarded as the dual counterpart to variation randomness.

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1. Introduction

The theory of algorithmic randomness gives precise mathematical meanings to the idea of a "random" real number. Among the various definitions, Martin-Löf randomness has been studied most thoroughly.

Solovay reducibility for left-c.e. reals is a method for comparing two reals in terms of their randomness. It enjoys several important properties—for example, it implies Turing reducibility and connects closely with Martin-Löf randomness.

During the 2000s, the field of algorithmic randomness developed rapidly. In the early 2000s, Downey, Hirschfeldt, and Nies [4] showed that the Solovay degrees of left-c.e. reals are dense; that is, between any two such degrees there is another one. This result led to many further discoveries from a degree-theoretic perspective.

The original notion of Solovay reducibility works well only for left-c.e. reals. More generally, we consider a hierarchy of four classes of real numbers, distinguished by how they can be approximated:

- Computable reals, which can be approximated effectively to any precision;
- Left-c.e. reals, which can be approximated from below by a non-decreasing computable sequence of rationals;
- Weakly computable reals, whose approximations may oscillate, but whose total variation is bounded;
- Computably approximable reals (also called c.a. reals or Δ^0_2 reals), which admit some computable converging sequence.

Zheng and Rettinger [15] extended Solovay reducibility to all c.a. reals, calling it S2a-reducibility. This agrees with the original notion of Solovay reducibility on left-c.e. reals and behaves well more generally. In our earlier work we simply called it Solovay reducibility; here we adopt "Solovay–Zheng–Rettinger reducibility" (Solovay–ZR reducibility) to avoid confusion.

In this paper, we ask whether Solovay–ZR degrees for all Δ^0_2 reals are dense. Recall that the Solovay degrees of left-c.e. reals form a dense partial order. However, this classical density argument heavily relies on the monotone nature of left-c.e. approximations. Once we move to the broader class of computably approximable reals, this monotonicity breaks down, and the original proof techniques cannot be directly generalized. The main contribution of this paper is to overcome this obstacle by introducing a novel approach based on Schnorr randomness, which allows us to establish a lower density in the Solovay–ZR degrees of c.a. reals. Here, "lower density" means that for every nontrivial degree, there always exists another degree strictly between it and the computable one.

Full density for all c.a. reals remains open. A natural next step is to examine whether our method can be refined to establish full density.

We also exhibit a quasi-minimal Solovay–ZR degree: a noncomputable weakly computable real whose only left-c.e. predecessors are actually computable. Thus, even though the overall Solovay–ZR hierarchy is lower dense, there can be a small "gap" among left-c.e. reals below a quasi-minimal degree.

In a different context, Miller [10] introduced the notion of variation randomness for weakly computable reals. We will show that this notion can be characterized by the condition that, in the Solovay–ZR degrees, there is no non-ML-random left-c.e. real above it. In other words, variation randomness and quasi-minimality are dual notions within the Solovay degrees.

Here is an outline of the paper. Section 3 explores the relationship between Solovay–ZR reducibility and total variation, leading to a characterization of variation randomness in terms of Solovay–ZR degrees. In Section 4, we prove the existence of quasi-minimal Solovay–ZR degrees among weakly computable reals relative to left-c.e. reals. This notion can be seen as the dual of variation randomness within the Solovay–ZR framework. Finally, Section 5 establishes that the Solovay–ZR degrees of computably approximable reals form a lower dense structure.

2. Preliminaries

For general background on algorithmic randomness, we refer the reader to [5]. In particular, Chapter 9 provides a detailed account of the Solovay degrees of left-c.e. reals.

2.1. **Computability of reals.** A function $f\colon \subseteq \omega \to \omega$ is said to be *partially computable* if it can be computed by a Turing machine. The notions of computability on the rationals $\mathbb{Q}$ and on sequences over $\mathbb{Q}$ are naturally induced.

A real $\alpha \in \mathbb{R}$ is called *computable* if there exists a computable sequence $(a_n)_{n\in\omega}$ of rationals such that $|\alpha - a_n| < 2^{-n}$ for all $n \in \omega$. A real $\alpha \in \mathbb{R}$ is *left-c.e.* if there exists an increasing computable sequence of rationals converging to α. A real $\alpha \in \mathbb{R}$ is *weakly computable* if there exists a computable sequence $(a_n)_n$ of rationals converging to α such that its variation $\sum_{n\in\omega} |a_{n+1}-a_n|$ is bounded. A real α is weakly computable if and only if it is the difference between two left-c.e. reals. Thus, weakly computable reals are also referred to as d.c.e. reals. A real $\alpha \in \mathbb{R}$ is *computably approximable* if there exists a computable sequence of rationals converging to α. The sets of all computable, left-c.e., weakly computable, and computably approximable reals are denoted by **EC**, **LC**, **WC**, and **CA**, respectively. With these definitions, we have the following hierarchy:

$$\mathbf{EC} \subsetneq \mathbf{LC} \subsetneq \mathbf{WC} \subsetneq \mathbf{CA}$$

2.2. **Martin-Löf randomness.** An open set $U \subseteq \mathbb{R}$ is called *c.e.* if there exists a computable sequence $((a_i, b_i))_i$ of open intervals with rational endpoints whose union is U. A Martin-Löf test (or ML-test) is a sequence $(U_n)_n$ of uniformly c.e. open sets in $\mathbb{R}$ such that $\mu(U_n) \leq 2^{-n}$ for all $n \in \omega$ where μ denotes the Lebesgue measure. A real α is said to be ML-random if $\alpha \notin \bigcap_n U_n$ for every ML-test $(U_n)_n$.

2.3. **Solovay reducibility.** The original definition of Solovay reducibility is as follows.

Definition 2.1 (Solovay [13])**.** For $\alpha, \beta \in \mathbb{R}$, α is Solovay reducible to β, denoted by $\alpha \leq_S \beta$, if there exists a partial computable function $f :\subseteq \mathbb{Q} \to \mathbb{Q}$ and a constant $c \in \omega$ such that $f(q) \downarrow$ for all $q < \beta$ and $\alpha - f(q) < c(\beta - q)$ for all $q \in \mathrm{dom}(f)$. Here, $f(q) \downarrow$ denotes that $f(q)$ is defined.

Intuitively, Solovay reducibility captures the idea that from the perspective of convergence rate, approximation of α can be slowed down by a constant multiple of that of β.

Within left-c.e. reals, Solovay reducibility admits the algebraic characterization as follows.

Theorem 2.2 (Downey, Hirschfeldt, and Nies [4])**.** *Let α, β be left-c.e. reals. Then, $\alpha \leq_S \beta$ if and only if there exist a constant $d \in \omega$ and a left-c.e. real γ such that*

$$d\beta = \alpha + \gamma.$$

The definition is modified by Zheng and Rettinger [15] as follows. They called it S2a-reducibility, but we will refer to it as Solovay–Zheng–Rettinger reducibility or simply Solovay–ZR reducibility.

Definition 2.3. For $\alpha, \beta \in \mathbf{CA}$, α is Solovay–ZR reducible to β, denoted by $\alpha \leq_{SZR} \beta$, if there exist computable sequences $(a_n)_{n\in\omega}$, and $(b_n)_{n\in\omega}$ converging to α, β, respectively, and a constant $c \in \omega$ such that

$$|\alpha - a_n| < c(|\beta - b_n| + 2^{-n}) \text{ for all } n \in \omega.$$

We note that Solovay reducibility and Solovay–ZR reducibility coincide on left-c.e. reals.

This relation is a preorder and naturally induces an equivalence relation. The resulting equivalence classes are called Solovay–ZR degrees.

Basic properties of Solovay–ZR reducibility were established in Zheng and Rettinger [15] and Rettinger and Zheng [12]. In particular, the least Solovay–ZR degree consists of all computable reals. Within the class of weakly computable reals, the maximal Solovay–ZR degree consists of all Martin-Löf random reals. Recall that a real number α is called *right-c.e.* if its negative $(-\alpha)$ is left-c.e. Since every real that is both weakly computable and Martin-Löf random is either left-c.e. or right-c.e.

([12]), it follows that the top Solovay–ZR degree among weakly computable reals is exactly the set of all left-c.e. or right-c.e. ML-random reals.

The Cauchy-style characterization of Solovay–ZR reducibility frequently turns out to be useful.

Theorem 2.4 (Kumabe, Miyabe, and Suzuki [8])**.** *Let $\alpha, \beta \in \mathbf{CA}$. The, $\alpha \leq_{SZR} \beta$ if and only if there exist computable sequences $(a_n)_n, (b_n)_n$ converging to α, β, respectively, and $q \in \omega$ such that*

$$(\forall k, n \in \omega)[k < n \Rightarrow |a_n - a_k| < q(|b_n - b_k| + 2^{-k})].$$

One of the rather non-obvious results is that Solovay reducibility among left-c.e. reals is dense. In fact, whenever α and β are left-c.e. reals with $\alpha <_S \beta$, there exists a left-c.e. real γ such that

$$\alpha <_S \gamma <_S \beta,$$

as shown by Downey, Hirschfeldt, and Nies [4].

3. Solovay–ZR reducibility and total variation

In this section, we discuss the relationship between Solovay–ZR reducibility and total variation. This relationship served as the initial motivation for our research.

Miller [10, Definition 3.3] introduced variation randomness. A weakly computable real $\alpha \in \mathbf{WC}$ is called *variation random* if, for every computable approximation $(a_n)_n$ of α whose variation is bounded, its variation

$$\sum_n |a_{n+1} - a_n|$$

is ML-random. Miller [10, Theorem 3.5] showed that there exists a weakly computable real that is non ML-random but is variation random. Miller [10, Proposition 3.4] also showed that being variation non-random is equivalent to being the difference of two non-ML-random left-c.e. reals. We can reformulate this with the terminology of Solovay–ZR reducibility as follows.

Proposition 3.1. *Let $\alpha \in \mathbf{WC}$ be a weakly computable real. Then the following conditions are equivalent:*

(i) *α is variation random.*
(ii) *For every left-c.e. real β, if $\alpha \leq_{SZR} \beta$, then β is ML-random.*

Proof. (ii)⇒(i). Let $(a_n)_n$ be a computable approximation of α with bounded variation γ. Then γ is clearly a left-c.e. real. Moreover, we have $\alpha \leq_{SZR} \gamma$ since

$$|\alpha - a_n| \leq \sum_{k=n}^{\infty} |a_{k+1} - a_k| = \gamma - \sum_{k=0}^{n-1} |a_{k+1} - a_k|$$

for all $n \in \omega$ and the last summation is a computable approximation of γ. Thus, by assumption (ii), γ is ML-random. Since the approximation $(a_n)_n$ is arbitrary, α is variation ML-random.

(i)⇒(ii). Suppose that condition (ii) fails. Then fix a left-c.e. real β such that $\alpha \leq_{SZR} \beta$ and β is not ML-random. By Theorem 2.4, there exist computable sequences $(a_n)_n$ and $(b_n)_n$ of rationals converging to α, β, respectively, and a constant $q \in \omega$ such that

$$(\forall k, n \in \omega)\,[k < n \Rightarrow |a_n - a_k| < q(|b_n - b_k| + 2^{-k})].$$

Suppose that $b_n < \beta$ holds for only finitely many n. Then β is right-c.e. Since β is also left-c.e., it follows that β is computable. This in turn implies that α is computable, because $\alpha \leq_{SZR} \beta$. Hence α is computable and condition (i) fails. Therefore, we may assume that $b_n < \beta$ for infinitely many n.

Since β is left-c.e., the relation $b_n < \beta$ is semi-decidable. Therefore, by passing to a subsequence if necessary, we may assume that $(b_n)_n$ is increasing. Hence, we have

$$|a_{n+1} - a_n| < q(b_{n+1} - b_n + 2^{-n}) \text{ for all } n \in \omega.$$

Let

$$c_n = q(b_{n+1} - b_n + 2^{-n}) - |a_{n+1} - a_n|.$$

By summing up, we have

$$\sum_{n\in\omega} |a_{n+1} - a_n| + \sum_{n\in\omega} c_n + qb_0 - 2q = q\beta.$$

Since $(c_n)_n$ is a computable sequence of positive rationals, its sum is left-c.e. Hence, by Theorem 2.2, the variation $\sum_n |a_{n+1} - a_n|$ is Solovay reducible to β. Since β is not ML-random, neither is the variation $\sum_n |a_{n+1} - a_n|$. Hence, α is not variation random. □

The proposition above states that a weakly computable real is variation random if and only if there is no non-ML-random left-c.e. Solovay–ZR degree strictly above it. Equivalently, among weakly computable reals, being both non-ML-random and variation random is precisely the same as having a Solovay–ZR degree that is *quasi-maximal* with respect to the left-c.e. reals–that is, above it there is no left-c.e. degree other than an ML-random one.

4. Quasi-minimal Solovay Degrees

In this section, we show the existence of quasi-minimal Solovay degrees.

Definition 4.1. A real $\beta \in \mathbf{CA}$ is *quasi-minimal relative to left-c.e. reals in the Solovay–ZR degrees*, or quasi-minimal for short, if

- β is not computable, and
- for every left-c.e. real α, $\alpha \leq_{SZR} \beta$ implies that α is computable.

Quasi-minimality is the dual notion of variation randomness among weakly computable reals. Recall that no left-c.e. real is quasi-minimal, since the partial order

of the Solovay degrees of left-c.e. reals is dense ([5, Theorem 9.5.3, Corollary 9.5.5]). The main claim of this section is the existence of such a non-computable real.

Theorem 4.2. *There exists a weakly computable real β that is quasi-minimal.*

We provide a proof of this theorem below.

4.1. **Requirements.** We assume $\beta \in (0,1)$. To ensure that β is not computable, we consider a computable list containing all computable reals in $[0,1]$, as follows. The e-th partially computable sequence of rationals in $[0,1]$ is denoted by $(x^e_s)_s$ and we assume that

$$|x^e_{s+1} - x^e_s| \leq 2^{-s-1} \text{ if both defined.}$$

We set the requirement R_e by

$$R_e : \ \beta \neq \lim_s x^e_s.$$

Then, satisfying all requirements R_e implies that β is not computable.

We also need to enforce the other condition of quasi-minimality, namely, if $\alpha \leq_{SZR} \beta$, then α is either computable or not left-c.e. We assume $\alpha \in [0,1]$. We computably enumerate all triples of two partially computable sequences of rationals in $[0,1]$ and a positive integer, each of which may serve as a potential witness for $\alpha \leq_{SZR} \beta$. That is, we enumerate all candidate approximations of α and β that may realize the relation $\alpha \leq_{SZR} \beta$. The sequences may be undefined from some point or may not be convergent.

To enforce that α is not left-c.e., we computably enumerate all nondecreasing sequences of rationals in $[0,1]$ such that every left-c.e. real in $[0,1]$ arises as the limit of some sequence in the enumeration. In fact, given a computable list of partial computable functions $\Phi_e : \omega \to \mathbb{Q} \cap [0,1]$, every left-c.e. real can be written as

$$\sup_{s,n\in\omega} \Phi_{e,s}(n),$$

where undefined values are interpreted as 0. Moreover, the approximations

$$y^e_s = \max_{t,n\leq s} \Phi_{e,t}(n)$$

yield the desired nondecreasing sequence.

We consider tuples of such sequences and integers, with e-th element denoted by $(a^e_s)_s$, $(b^e_s)_s$, q_e, $(y^e_s)_s$. Note that y^e_s is defined for all $e, s \in \omega$. We set the requirement Q_e by

$$\begin{aligned} Q_e :&(a^e_s)_s \in \mathrm{CS}(\alpha), (b^e_s)_s \in \mathrm{CS}(\beta), (\forall s)[|\alpha - a^e_s| < q_e|\beta - b^e_s| + 2^{-s}] \\ &\Rightarrow \alpha \text{ is computable or } \alpha \neq \lim_s y^e_s. \end{aligned}$$

Here, $\mathrm{CS}(\alpha)$ is the set of all computable sequences of rationals converging to α. Then, satisfying all requirements Q_e ensures the second condition of quasi-minimality of β.

4.2. **Local strategy.** First, we present a strategy to satisfy Q_e for a fixed $e \in \omega$. We search for two approximations $a^e_{s_0}$ and $a^e_{s_1}$ that are sufficiently far apart. If they cannot be found and a^e_s is defined for all $s \in \omega$, then α should be computable. If found, then, by choosing β appropriately, one can make α close to whichever of $a^e_{s_0}$ or $a^e_{s_1}$ one prefers. Finally, by choosing β in accordance with the monotonically increasing values of y^e_s, we can ensure that α does not coincide with its limit. We choose β so that α lies in the right-hand interval while y^e_s is small, and in the left-hand interval once y^e_s becomes large.

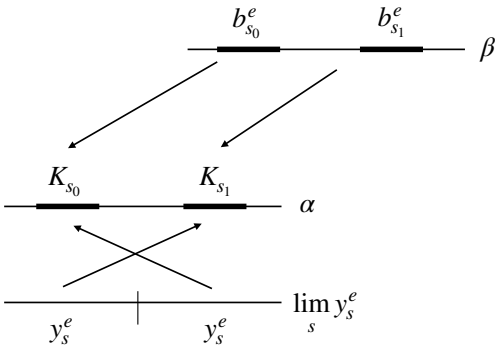

FIGURE 1. Local strategy

We now describe the construction in more concrete terms. Let K^e_s be the closed interval given by

$$K^e_s = [a^e_s - 2^{-s+1}, a^e_s + 2^{-s+1}].$$

Search for $s_0, s_1 \in \omega$ such that $K^e_{s_0}$ and $K^e_{s_1}$ are disjoint. If K^e_s is undefined for some $s \in \omega$, then the requirement Q_e is satisfied. If K^e_s is defined for all $s \in \omega$, and if no disjoint pair of intervals can be found (that is, every two intervals intersect), then $(a^e_s)_s$ is a Cauchy sequence and its limit is computable, which implies that the requirement Q_e is satisfied. Thus, we assume that such a pair s_0 and s_1 is found.

We further assume that $K^e_{s_0}$ lies entirely to the left of $K^e_{s_1}$. More precisely, for any $y_0 \in K^e_{s_0}$ and $y_1 \in K^e_{s_1}$, we have $y_0 < y_1$. First, we enforce that β is sufficiently close to $b^e_{s_1}$ so that α should be in $K^e_{s_1}$. Note that y^e_s is increasing in s. If y^e_s gets close to $K^e_{s_1}$, then we enforce that β is sufficiently close to $b^e_{s_0}$ so that α lies in $K^e_{s_0}$. In any case, α is not the limit of y^e_s, and the requirement Q_e is satisfied.

4.3. **Combined strategy.** We set the priorities of the requirements as follows:

$$Q_0 > R_0 > Q_1 > R_1 > \cdots .$$

Each requirement is associated with a closed forcing interval: I^e for Q_e and J^e for R_e. Each interval associated with a requirement with lower priority should be included in each interval associated with a requirement with higher priority, that is,

$$I^0 \supseteq J^0 \supseteq I^1 \supseteq J^1 \cdots .$$

Initially (i.e., at stage $s = 0$), all intervals are set to $[0, 1]$. For convenience, let $J^{-1} = [0, 1]$.

We describe a strategy to satisfy Q_e. Let $s \in \omega$ be the current stage. Search for $s_0, s_1 \in \omega$ such that

(i) $a^e_{s_0}$, $b^e_{s_0}$, $a^e_{s_1}$, and $b^e_{s_1}$ are defined,
(ii) $b^e_{s_0}$ and $b^e_{s_1}$ are interior points of J^{e-1},
(iii) $a^e_{s_0} < a^e_{s_1}$, and
(iv) $K^e_{s_0}$ and $K^e_{s_1}$ are disjoint.

Until such a pair is found, set $I^e = J^{e-1}$. As long as J^{e-1} remains unchanged, we use a fixed pair (s_0, s_1). We choose the pair with the smallest possible larger index $\max\{s_0, s_1\}$, and among such pairs, the one with the smallest smaller index $\min\{s_0, s_1\}$. Let $z_e = (a^e_{s_0} + 2^{-s_0+1} + a^e_{s_1} - 2^{-s_1+1})/2$, which is the midpoint between the right endpoint of K_{s_0} and the left endpoint of K_{s_1}. Since K_{s_0} and K_{s_1} are disjoint, we have $z_e \notin K_{s_0} \cup K_{s_1}$. Recall that s is the current stage. While $y^e_s < z_e$, we set

$$I^e = [b^e_{s_1} - 2^{-s_1} q_e^{-1}, b^e_{s_1} + 2^{-s_1} q_e^{-1}] \cap J^{e-1}.$$

If $y^e_s \geq z_e$, then set

$$I^e = [b^e_{s_0} - 2^{-s_0} q_e^{-1}, b^e_{s_0} + 2^{-s_0} q_e^{-1}] \cap J^{e-1}.$$

We describe a strategy to satisfy R_e. Search $s_2 \in \omega$ such that

(i) $x^e_{s_2}$ is defined, and
(ii) 2^{-s_2+1} is smaller than r times the length of I^e,

where $r > 0$ is a positive rational; in fact $r < 1/6$ is sufficient as we will see later. Until such an s_2 is found, we define J^e to satisfy only conditions (i) and (iii) below. As long as I^e remains unchanged, we use a fixed integer s_2. Let J^e be a closed interval such that

(i) $J^e \subseteq (I^e)^\circ$,
(ii) J^e and $[x^e_{s_2} - 2^{-s_2}, x^e_{s_2} + 2^{-s_2}]$ are disjoint, and
(iii) the length of J^e is less than r times the length of I^e,

where I° is the set of all inner points of I.

Finally, define β as the unique real in the intersection $\bigcap I_e \cap \bigcap J_e$.

4.4. **Verification.** We will establish several claims on the construction.

Claim. *Each requirement is injured only finitely many times.*

As long as the requirement with higher priority remain unchanged, the forcing intervals change only finitely many times. Thus, the claim follows by induction.

Claim. *The real β is well-defined.*

The forcing intervals are closed and nested. Their lengths decrease to 0 because of the condition (iii) in the definition of J^e. Hence, the intersection consists of a single real.

Claim. *Each requirement R_e is satisfied.*

By condition (ii) in the definition of J^e, we have

$$|\beta - x^e_{s_2}| > 2^{-s_2}.$$

By the fast convergence of $(x^e_s)_s$, if $x^e = \lim_s x^e_s$ exists, then we have

$$|x^e - x^e_{s_2}| \leq 2^{-s_2}.$$

Thus, the requirement R_e is satisfied.

Claim. *Each requirement Q_e is satisfied.*

Fix e. Suppose that $(a^e_s)_s \in \mathrm{CS}(\alpha)$, $(b^e_s)_s \in \mathrm{CS}(\beta)$, and $|\alpha - a^e_s| < q_e|\beta - b^e_s| + 2^{-s}$ for all $s \in \omega$. After fixing J^{e-1}, we have $\beta \in J^e \subseteq (I^e)^o$ and $I^e \subseteq J^{e-1}$, thus β is an interior point of J^{e-1}. Therefore, we have $b^e_s \in (J^{e-1})^o$ for all sufficiently large s. Hence, conditions (i) and (ii) in the strategy for Q_e are satisfied for all sufficiently large s. Thus, if no such pair (s_0, s_1) is found, then for all sufficeintly large s condition (iv) fails. In other words, every two intervals K^e_s intersect, and α must be computable in the same way as the local strategy in Section 4.2. In the following, we look at the case when such a pair exists.

If $y^e_s < z_e$ for all s, then $\beta \in I^e$, and $|\beta - b^e_{s_1}| \leq 2^{-s_1} q_e^{-1}$ by the definition of I^e, and thus

$$|\alpha - a^e_{s_1}| < q_e|\beta - b^e_{s_1}| + 2^{-s_1} \leq 2^{-s_1+1},$$

which implies that $\alpha \neq \lim_s y^e_s$, because

$$\lim_s y^e_s \leq z_e < a^e_{s_1} - 2^{-s_1+1} < \alpha.$$

If $y^e_s > z_e$ for some s, then $y^e_s > z_e$ for all but finitely many s since $(y^e_s)_s$ is non-decreasing. Thus,

$$|\alpha - a^e_{s_0}| < q_e|\beta - b^e_{s_0}| + 2^{-s_1} \leq 2^{-s_0+1},$$

which implies that

$$\alpha < a^e_{s_0} + 2^{-s_0+1} < z_e \leq \lim_s y^e_s.$$

Thus, the requirement Q_e is satisfied.

Claim. *The real β is weakly computable.*

We analyze the number of injuries associated with Q_e and R_e after fixing requirements with higher priority. The requirement Q_e has the following stages:

(i) Q_e is in the stage of searching for s_0, s_1.
(ii) I^e is defined near $b^e_{s_1}$.

(iii) I^e is defined near $b^e_{s_0}$.

Thus, the number of changes mainly associated with Q_e is 3.

The requirement R_e has the following stages:

(i) R_e is in the stage of searching for s_2.
(ii) J^e is defined near $x^e_{s_2}$.

Thus, the number of changes associated mainly with R_e is 2. The total number of changes is:

$$Q_0 : 3, \quad R_0 : 3 \times 2 = 6, \quad Q_1 : 6 \times 3, \quad R_1 : 6^2.$$

For all $e \in \omega$, the total number of changes of Q_e and R_e is bounded by 6^{e+1}.

We claim that, by choosing r sufficiently small, β becomes weakly computable. When I^e or J^e is redefined, all intervals associated with requirements with lower priorities are reset. As an approximation of β, we use the midpoint of I^e or J^e, whichever is the narrowest among the defined intervals. The length of the interval I^e is bounded by r^e. When the interval J^e is modified by requirement R_e, both the old and new midpoints lie within I^e, so the resulting approximation error is also bounded by r^e. Similarly, the length of the interval J^{e-1} is bounded by r^e, and when I^e is altered by requirement Q^e, the approximation difference remains bounded by r^e. Thus, the total sum of the approximation differences is bounded by

$$2\sum_{e=0}^{\infty} 6^{e+1} \cdot r^e,$$

which is finite provided r is sufficiently small.

5. Lower density of Solovay–ZR degrees

We establish the lower density of Solovay–ZR degrees within the class of computably approximable reals. The proof by Downey, Hirschfeldt, and Nies [4] establishing the lower density of Solovay degrees for left-c.e. reals relies critically on the monotonicity of their approximations, and therefore cannot be directly adapted to our setting.

Interestingly, our argument splits into cases depending on whether the sequence is Schnorr random or not. Intuitively, if a real is random, then by applying a suitable Bernoulli measure transformation one can make it slightly less random in the sense of Solovay reducibility. In this process, the result on the longest run ensures a close correspondence between (i) the distance between two reals and (ii) the length of the common prefix of their binary expansions, so that one can pass from one description to the other without large discrepancies.

On the other hand, if a real is not random, then one can construct a function with a steep slope at that point, which allows us to produce a new real that is even less random. To ensure the computability of the transformation, we rely on

Schnorr randomness, which imposes stricter computability constraints than Martin-Löf randomness.

5.1. **Schnorr randomness and the longest run of heads.** We quickly review Schnorr randomness. We identify a real in $[0,1]$ with its infinite binary expansion, and call such a real Schnorr random with respect to the Lebesgue measure if its binary expansion is Schnorr random with respect to the uniform measure. The Cantor space 2^ω is the set of all infinite binary sequences, equipped with the topology generated by the cylinder sets $[\sigma] = \{X \in 2^\omega \ : \ \sigma \prec X\}$, where $\prec$ denotes the prefix relation. A c.e. open set on 2^ω is the union of a computable sequence of cylinder sets. Let μ be the uniform measure on 2^ω. A *Schnorr test* is a sequence $(U_n)_{n\in\omega}$ of uniformly c.e. open sets such that $\mu(U_n) \leq 2^{-n}$ for all $n \in \omega$, with the additional requirement that $\mu(U_n)$ is uniformly computable. A sequence X is called *Schnorr random* if $X \notin \bigcap_n U_n$ for any Schnorr test $(U_n)_n$.

A useful fact on Schnorr randomness is a Solovay-test-type characterization by Downey and Griffiths [3, Definition 2.3, Theorem 2.4]. A *total Solovay test* is a computable collection of c.e. open sets $\{V_i\}_{i\in\omega}$ such that the sum $\sum_{i=0}^{\infty} \mu(V_i)$ converges to a computable real. A sequence $X \in 2^\omega$ passes a total Solovay test if $X \in V_i$ for at most finitely many V_i. Then, $X \in 2^\omega$ is Schnorr random if and only if X passes all total Solovay tests.

In the proof below, we consider the longest run of heads in fair coin tosses. The relevance of the longest run result lies in its ability to connect probabilistic behavior of random sequences with metric properties of real numbers, a connection we exploit in the density argument.

A survey of unpublished work on this topic can be found in [1] though we require only a much weaker bound. Accordingly, we invoke Durrett's example in [6, Example 2.3.12], which supplies a proof of the required bound. The following is a straightforward reformulation of this result in the terminology of algorithmic randomness. A related result for a string with high Kolmogorov complexity can be found in [9].

Theorem 5.1. *For $X \in 2^\omega$, define $\ell_n(X) = \max\{m \ : \ X_{n-m+1} = \cdots = X_n = 1\}$, where X_n denotes the n-th bit of $X \in 2^\omega$. Let $L_n(X) = \max_{1\leq m\leq n} \ell_m$. Then $L_n(X)$ is referred to as the longest run up to time n. If X is Schnorr random with respect to the uniform measure μ, then*

$$\frac{L_n(X)}{\log_2 n} \to 1, \ (n \to \infty).$$

Proof. Let $\varepsilon > 0$ be a rational number. For all $n \geq 1$, let

$$U_n = \{X \in 2^\omega \ : \ \ell_n(X) \geq (1+\varepsilon)\log_2 n\}.$$

Then, we have $\mu(U_n) \leq n^{-(1+\varepsilon)}$ and $\mu(U_n)$ is computable uniformly in n. Furthermore, we have

$$\sum_{n=N+1}^{\infty} \mu(U_n) \leq \sum_{n=N+1}^{\infty} n^{-(1+\varepsilon)} \leq \int_N^{\infty} x^{-(1+\varepsilon)} \, \mathrm{d}x = \frac{N^{-\varepsilon}}{\varepsilon},$$

which converges to 0 as $N \to \infty$. Hence, $(U_n)_n$ is a total Solovay test. Since X is Schnorr random, $X \in U_n$ for at most finitely many U_n. Since ε is arbitrary, we have

$$\limsup_n \frac{L_n(X)}{\log_2 n} \leq 1.$$

For the other direction, suppose that ε is small enough. We break the first n bits into disjoint blocks of length $[(1-\varepsilon)\log_2 n] + 1$. For sufficiently large n, there are at least $\frac{n}{\log_2 n}$ such blocks. For $n \geq 1$, let

$$V_n = \{X \in 2^{\omega} \ : \ L_n(X) \leq (1-\varepsilon)\log_2 n\}.$$

Clearly, $\mu(V_n)$ is computable uniformly in n. For each $X \in V_n$, in each block, not all bits are equal to 1. Thus,

$$\mu(V_n) \leq (1 - 2^{-[(1-\varepsilon)\log_2 n]-1})^{n/\log_2 n} \leq \exp(-n^{\varepsilon}/2\log_2 n) \leq n^{-2},$$

where the details can be found in [6, Example 2.3.12]. Therefore, we have

$$\sum_{n=N+1}^{\infty} \mu(V_n) \leq \sum_{n=N+1}^{\infty} n^{-2} \leq \int_N^{\infty} x^{-2} \, \mathrm{d}x = N^{-1}.$$

Hence, $(V_n)_n$ is a total Solovay test. Since X is Schnorr random and ε is arbitrary, we obtain

$$\liminf_n \frac{L_n(X)}{\log_2 n} \geq 1.$$

□

This leads to a result concerning the relationship between the distance of two real numbers and the number of initial matching digits in their binary expansions.

Lemma 5.2. *Let $X, Y \in 2^{\omega}$ and x, y be the corresponding reals in $[0,1]$, that is, $x = \sum_{k=0}^{\infty} X(k)2^{-k-1}$, $y = \sum_{k=0}^{\infty} Y(k)2^{-k-1}$. Suppose $x < y$. Let $n = \max\{k \in \omega \ : \ X \upharpoonright k = Y \upharpoonright k\}$ and $d = -\log_2(y-x)$, where $X \upharpoonright k$ denotes the initial segment of X with length k. Then,*

$$X(n) = 0, \text{ and } X(k) = 1 \text{ for all } k \text{ such that } n < k < [d],$$
$$Y(n) = 1, \text{ and } Y(k) = 0 \text{ for all } k \text{ such that } n < k < [d].$$

Proof. Let $Z \in 2^{\omega}$ be defined by

$$Z \upharpoonright n = Y \upharpoonright n, \ Z(n) = 1, \text{ and } Z(k) = 0 \text{ for all } k > n,$$

and $z \in [0,1]$ be the corresponding real. Since $X(n) \neq Y(n)$ and $x < y$, it follows that $X(n) = 0$ and $Y(n) = 1$. Thus, $x \leq z \leq y$.

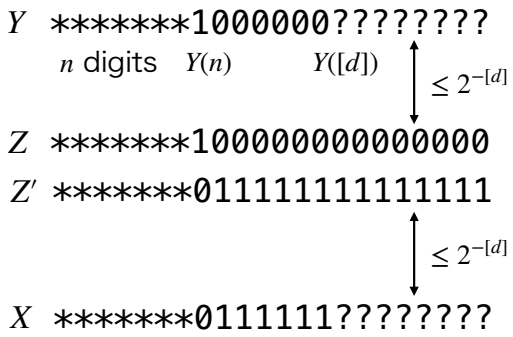

FIGURE 2. Runs of consecutive 0s and 1s

Since $y - z \leq y - x = 2^{-d} \leq 2^{-[d]}$, we have

$$Y(k) = Z(k)$$

for all k such that $k < [d]$; otherwise the contribution of that differing bit would exceed $2^{-[d]}$.

Let $Z' \in 2^\omega$ be such that

$$Z' \upharpoonright n = X \upharpoonright n,\ Z'(n) = 0, \text{ and } Z'(k) = 1 \text{ for all } k \geq n.$$

Then, the corresponding real of Z' is also z. Since $z - x \leq y - x \leq 2^{-[d]}$, we also have

$$X(k) = Z'(k) \quad \text{for all } k < [d].$$

□

5.2. **The case of Schnorr random reals.** To establish the lower density of Solovay–ZR degrees among computably approximable reals, as a first step, we consider the case where β is Schnorr random.

In the proof below, we employ the following notion from [7].

Definition 5.3. Let $\alpha, \beta \in \mathbf{CA}$. The real α is *strongly Solovay reducible* to β, denoted by $\alpha \ll_S \beta$, if there exist computable sequences $(a_n)_n$ and $(b_n)_n$ of rationals converging to α and β, respectively, such that

$$\lim_{n\to\infty} \frac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}} = 0. \tag{1}$$

Strong Solovay reducibility is a stronger version of Solovay–ZR reducibility; The convergence rate of α should be faster than that of β. It may be more precise to refer to this notion as *strong Solovay–ZR reducibility*. It is evident that strong Solovay reducibility implies Solovay–ZR reducibility. Moreover, for any computably approximable real number α, the condition $\alpha \ll_S \alpha$ holds if and only if α is computable, as shown in [7, Proposition 3.10].

Theorem 5.4. *Let $\beta \in \mathbf{CA}$ be a Schnorr random real. Then there exists $\alpha \in \mathbf{CA}$ such that $\emptyset <_{SZR} \alpha <_{SZR} \beta$. If $\beta \in \mathbf{WC}$, then such an α can be chosen to lie in $\mathbf{WC}$ as well.*

Proof. Let $p \in \left(\frac{1}{2}, 1\right)$ be a computable real number. Let B_p denote the Bernoulli measure on 2^ω with parameter p (that is, the measure under which each bit takes the value 1 independently with probability p). We identify 2^ω with the unit interval $[0,1]$ via the usual binary expansion. Define a function $f : [0,1] \to [0,1]$ by

$$f(x) = B_p\Big(\{z \in [0,1] : z \leq x\}\Big),$$

where the relation $\leq$ is the standard order on the real numbers. Then the following properties hold:

(i) The function f is computable (in the sense of computable analysis; see, e.g., [14, 2]).
(ii) The function f is strictly monotone increasing.

We then define $\alpha = f(\beta)$. Since $\alpha \equiv_T \beta$, it follows that α is not computable.

It remains to show that α is strongly Solovay reducible to β. First, note that $\alpha \ll_S \beta$ implies $\alpha \leq_{SZR} \beta$. If $\beta \leq_{SZR} \alpha$, then $\alpha \ll_S \beta \leq_{SZR} \alpha$, which would imply $\alpha \ll_S \alpha$, a contradiction.

Fix $(b_n)_n \in \mathrm{CS}(\beta)$ and let $a_n = f(b_n)$ for all n. Although a_n may not be rational, it suffices to verify condition (1) for this sequence $(a_n)_n$, since rational approximations can be obtained by a small adjustment.

Fix $n \in \omega$. Let $\gamma \in 2^\omega$ be a binary expansion of the rational number b_n. Since β is Schnorr random and b_n is rational, we have $\beta \neq b_n$. Define

$$m = \max\{k \in \omega \ : \ \beta \upharpoonright k = \gamma \upharpoonright k\}, \quad d = -\log_2 |\beta - b_n|.$$

We assume that $\beta - b_n$ is sufficiently small and that $[d] \geq 2$ for all $n \in \omega$. By Lemma 5.2, the difference $d - m$ is bounded by the length of the longest run of 0s or 1s up to time n. Since β is Schnorr random, it follows from Theorem 5.1 that there exists a constant $c \in \omega$ such that

$$d - m \leq c \log_2[d] \quad \text{for all } n \in \omega. \tag{2}$$

Note that the constant $c \in \omega$ depends on β, but not on $n \in \omega$. Also note that both d and m depend on $n \in \omega$.

Let I be the closed interval between b_n and β. To verify condition (1), it suffices to show that

$$\frac{B_p(I)}{2^{-d}} \to 0 \tag{3}$$

as $n \to \infty$ because

$$|\beta - b_n| = 2^{-d}, \quad |\alpha - a_n| = B_p(I).$$

As $n \to \infty$, we have $m \to \infty$ and $d \to \infty$. Thus, condition (3) is equivalent to

$$\log_2 B_p(I) + d \to -\infty,$$

which is implied by

$$\limsup_{n\to\infty} \left(\frac{\log_2 B_p(I)}{d} + 1 \right) < 0.$$

Now, we evaluate this value. Since $I \subseteq [\beta \upharpoonright m]$, we have

$$B_p(I) \leq B_p([\beta \upharpoonright m]) = p^{\ell_1} q^{\ell_0}$$

where ℓ_1 and ℓ_0 denote the number of 1s and 0s, respectively, in $\beta \upharpoonright m$, and $q = 1-p$. By taking the base-2 logarithm and dividing by d, we have

$$\frac{\log_2 B_p(I)}{d} \leq \frac{\ell_1}{d} \log_2 p + \frac{\ell_0}{d} \log_2 q.$$

Hence, it suffices to show that the lim sup of the following expression is strictly less than 0:

$$1 + \frac{\ell_1}{d} \log_2 p + \frac{\ell_0}{d} \log_2 q \tag{4}$$

Now we use the randomness. By the effective law of large numbers for Schnorr randomness, together with the bound (2), we obtain

$$\frac{d-m}{d} \to 0, \quad \frac{\ell_1}{d} = \frac{\ell_1}{m}\left(1 - \frac{d-m}{d}\right) \to \frac{1}{2}, \quad \frac{\ell_0}{d} \to \frac{1}{2}.$$

A straightforward calculation shows that for $0 < x < 1$,

$$-\ln x - \ln(1-x) \;\geq\; 2\ln 2,$$

with equality at $x = \frac{1}{2}$, which implies

$$1 + \frac{1}{2}\log_2 p + \frac{1}{2}\log_2 q < 0.$$

Hence, the limsup of (4) is less than 0.

Now, since we have $\alpha \leq_{SZR} \beta$, by means of Theorem 2.4, it holds that $\beta \in \mathbf{WC} \implies \alpha \in \mathbf{WC}$. □

5.3. **The case of non-Schnorr-random reals.** We next consider the case where β is not Schnorr random, which completes the proof of the lower density of Solovay–ZR degrees among computably approximable reals.

In the proof below, we use Schnorr integral tests from [11]. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ and μ be the Lebesgue measure on $[0,1]$. A *Schnorr integral test* is a lower semi-computable function $f : [0,1] \to \overline{\mathbb{R}}$ such that the integral $\int f \, \mathrm{d}\mu$ is finite and a computable real. A real $x \in [0,1]$ is Schnorr random if and only if $f(x) < \infty$ for every Schnorr integral test f.

Theorem 5.5. *Let $\beta \in \mathbf{CA}$ be a noncomputable real that is not Schnorr random. Then there exists $\alpha \in \mathbf{CA}$ such that $\emptyset <_{SZR} \alpha <_{SZR} \beta$. If $\beta \in \mathbf{WC}$, then such an α can be chosen to lie in $\mathbf{WC}$ as well.*

Proof. Suppose that β is noncomputable and not Schnorr random. Then there exists a lower semicomputable function $f : [0,1] \to \overline{\mathbb{R}}$ such that $I = \int f \, \mathrm{d}\mu$ is a computable real and $f(\beta) = \infty$. We further assume that $f \geq \frac{1}{2}$ and $I = 1$.

Let $g = \frac{1}{f}$. Then, g is an upper semicomputable function $g : [0,1] \to [0,2]$ such that the integral is a positive computable real. Thus, there exists a positive computable real c such that the integral of $\hat{g} = c \cdot g$ equals 1. Notice that $\hat{g}(x) = 0$ if and only if $f(x) = \infty$. In particular, we have $\hat{g}(\beta) = 0$.

Let $h(x) = \int_0^x \hat{g}(t) \, \mathrm{d}\mu$. Then h is a computable function from $[0,1]$ to $[0,1]$, and is Lipschitz continuous.

We define α to be $h(\beta)$. Then we clearly have $\alpha \leq_{SZR} \beta$; see [8] for the connection between Solovay–ZR reducibility and Lipschitz continuity. Since h is strictly increasing and computable, the inverse h^{-1} is also computable (see [14, Theorem 6.3.11]), and thus $\alpha \equiv_T \beta$. In particular, α is not computable.

Finally, we claim that $\alpha \ll_S \beta$, which implies that $\beta \not\leq_S \alpha$. Let $(b_n)_n \in \mathrm{CS}(\beta)$. Since f is lower semicomputable and $f(\beta) = \infty$, we may further assume that $f(b_n) > n$. Define a_n to be a rational number such that $|a_n - h(b_n)| < 2^{-2n}$. We may further assume that $(a_n)_n$ is computable, so that $(a_n)_n \in \mathrm{CS}(\alpha)$. It now suffices to show that

$$\frac{|\alpha - a_n|}{|\beta - b_n| + 2^{-n}} \to 0 \text{ as } n \to \infty.$$

Let $M > 0$ be an arbitrarily large integer. Then there exists an open interval I containing β such that $f(x) > M/c$ for all $x \in I$. Since $g(x) < c/M$ and $\hat{g}(x) < M^{-1}$ for all $x \in I$, the function h restricted to I is Lipschitz continuous with Lipschitz constant M^{-1}. Since $b_n \to \beta$ as $n \to \infty$, there exists $N \in \omega$ such that $f(x) > M/c$ for all reals x between β and b_n, for all $n \geq N$. Then,

$$|\alpha - a_n| \leq |h(\beta) - h(b_n)| + 2^{-2n} \leq M^{-1}|\beta - b_n| + 2^{-2n},$$

which implies the desired claim.

If $\beta \in \mathbf{WC}$, then we obtain $\alpha \in \mathbf{WC}$ because $\alpha \leq_{SZR} \beta$. □

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