Randomness with respect to c.e. semimeasures

Kenshi Miyabe

[a]*Meiji University, Department of Mathematics, 1-1-1 Higashi-Mita, Tama-ku, Kawasaki-shi, 214-8571, Kanagawa, Japan*

Abstract

We study algorithmic randomness with respect to c.e. semimeasures, which naturally arise as pushforward measures of partial computable mappings and therefore play a crucial role in algorithmic randomness. We consider four distinct randomness notions: three based on complexity and one based on tests. We systematically clarify their inclusion relationships. Our main contribution is to construct concrete examples that separate these notions. Furthermore, we investigate how they interact with the classical randomness preservation and no-randomness-from-nothing theorems, identifying precise conditions under which they continue to hold.

Keywords: Martin-Löf randomness, partial randomness, Kolmogorov complexity
2020 MSC: 03D32, 68Q30

1. Introduction

Let us consider the Cantor space and a computable measure on it. What does it mean for a point in this measure space to be random? Martin-Löf randomness is one of the natural definitions of such randomness by tests and it is characterized robustly through Kolmogorov complexity and martingales.

Recently, randomness with respect to c.e. semimeasures has started to attract attention. Bienvenu et al. [1] appears to be the first study in this area, which focused on *randomness preservation* and *no-randomness-from-nothing*. The randomness preservation theorem states that for a computable map defined almost everywhere, a random point is mapped to a random point with respect to the pushforward measure. The no-randomness-from-nothing

Email address: research@kenshi.miyabe.name (Kenshi Miyabe)

theorem asserts that, given a computable map defined almost everywhere, any random point in the target space has a preimage containing a random point in the original space. Consequently, randomness with respect to a computable measure is preserved under computable mappings defined almost everywhere.

In general, the pushforward measure of a partial computable mapping is a c.e. semimeasure. How, then, should we define randomness with respect to a c.e. semimeasure? Could it correspond to images of random sequences under partial computable mappings, as described for computable measures? This has a negative answer: there exist two partial computable functions with the same pushforward measure but differing images of random sequences as shown in Bienvenu et al. [1]. In this sense, a general extension does not hold. We clarify the situation in this paper by defining several concepts of randomness and examining the hierarchical structure among them.

A related paper by Barmpalias and Shen [2] explores a notion of randomness for c.e. semimeasures different from that proposed by Bienvenu et al. [1].

In this paper we address the following questions:

Q1 How should one define randomness with respect to an arbitrary c.e. semimeasure?

Q2 Do classical theorems such as randomness preservation and no-randomness-from-nothing extend to these notions?

Let $f : 2^{<\omega} \to [0, \infty]$ be a function, which corresponds to a c.e. semimeasure μ by $f(\sigma) = -\log \mu(\sigma)$. To investigate algorithmic randomness relative to μ, we consider the following four notions, which are motivated by analogous distinctions appearing in the study of partial randomness.

(I) KA-f-complexity: $\mathrm{KA}(X \upharpoonright n) > f(X \upharpoonright n) - O(1)$.
(II) Strong K-f-complexity: $K(X \upharpoonright n) - f(X \upharpoonright n) \to \infty$ as $n \to \infty$.
(III) K-f-complexity: $K(\upharpoonright n) > f(X \upharpoonright n) - O(1)$.
(IV) f-ML-randomness: The definition is by tests.

The test-based notion (IV) was introduced and studied by Bienvenu et al. [1], who developed a general framework of Martin-Löf randomness with respect to c.e. semimeasures and established the randomness preservation property. Building on this framework, Barmpalias and Shen [2] investigated the KA-based definition (I) (under a different terminology) and proved no-randomness-from-nothing under a suitable restriction on oracle use.

While these studies addressed specific aspects of randomness with respect to c.e. semimeasures, a systematic comparison among multiple complexity-based definitions has not previously been undertaken. The present paper fills this gap by introducing the four notions above and showing that they form a proper hierarchy:

$$(\mathrm{I}) \Rightarrow (\mathrm{II}) \Rightarrow (\mathrm{III}) \Rightarrow (\mathrm{IV}),$$

and each implication is strict. This unifies and extends the perspectives of Bienvenu et al. [1] and Barmpalias and Shen [2] by placing both the test-based and complexity-based approaches within a single coherent framework, and by clarifying the precise relationships among these four notions, answering Q1.

Furthermore, although the study of randomness with respect to c.e. semimeasures has so far been developed independently of the theory of partial randomness, we observe that these two areas share striking conceptual similarities. In particular, the hierarchy of complexity-based and test-based randomness notions established in the study of partial randomness (see Hudelson's doctoral thesis [3] for a comprehensive account) provides a useful perspective for analyzing randomness relative to c.e. semimeasures. By bringing these notions into the present context, we obtain a natural framework for comparing different definitions of randomness with respect to semimeasures. The resulting hierarchy, however, is not identical to that of partial randomness. This connection to partial randomness, and the clarification of how the two hierarchies differ, constitute one of the contributions of this paper. A summary of the related concepts and comparisons is provided in Section 2.

We also show for which randomness notions randomness preservation and no-randomness-from-nothing results hold, answering Q2. A more detailed explanation will be provided in Section 2.

In Section 3 we will show that all notions from (I) to (IV) are equivalent for a computable and concave function f (Corollary 3.4). In Section 4 we will show the implications, and in Section 5 we will show the failure of reverse implications for an upper semicomputable and concave function f. In Section 6 we will study randomness preservation and no-randomness-from-nothing results for the notions defined in this paper.

2. Definitions, related results, and contributions

Basic references in algorithmic randomness include Nies [4], Downey and Hirschfeldt [5], and Shen, Uspensky, and Vereshchagin [6].

2.1. Preliminaries

In this subsection, we recall some basic definitions and notations. We denote by $2^{<\omega}$ the set of all finite binary strings. Cantor space 2^ω is the set of all infinite binary sequences, with the topology generated by cylinder sets $[\sigma] = \{X \in 2^\omega \ : \ \sigma \prec X\}$, where $\prec$ is the prefix relation. For a set A of strings, let $[A] = \bigcup_{\sigma\in A}[\sigma]$. A (probability) measure μ on the Cantor space is uniquely determined by its values on the cylinder sets. Hence, it can be regarded as a function $\mu : 2^{<\omega} \to [0, 1]$. A measure μ is called *computable* when $\mu(\sigma)$ is a computable real uniformly in σ.

A *semimeasure* μ on the Cantor space is a function from $2^{<\omega}$ to $[0, 1]$ such that

$$\mu(\sigma) \geq \mu(\sigma 0) + \mu(\sigma 1) \text{ for all } \sigma \in 2^{<\omega}.$$

A semimeasure μ is c.e. (or computably enumerable) if it is a lower semicomputable function.

For two semimeasures μ and ν, we say that μ *dominates* ν if there exists a constant $c \in \omega$ such that $\nu(\sigma) \leq c\,\mu(\sigma)$ for all $\sigma \in 2^{<\omega}$. A c.e. semimeasure μ is called *optimal* if it dominates all c.e. semimeasures. A *priori complexity* $\mathrm{KA}(\sigma)$ of strings $\sigma \in 2^{<\omega}$ is defined by $-\log \mu(\sigma)$ with respect to an optimal c.e. semimeasure μ where the base of log is 2. The *prefix-free complexity* $K(\sigma)$ is the length of the shortest input of an optimal prefix-free Turing machine that outputs σ:

$$K(\sigma) = \min\{|\tau| \ : \ U(\tau) = \sigma\}.$$

By the optimality, KA and K are both well-defined up to a constant. Both complexities indicate how complex the input strings are. It can be shown that $\mathrm{KA}(\sigma) \leq K(\sigma) + O(1)$.

A c.e. open set U on the Cantor space is an open set that can be written as $U = \bigcup_{\sigma\in S}[\sigma]$ where S is a c.e. (or equivalently computable) set of strings in $2^{<\omega}$. For a computable measure μ, a *Martin-Löf test* with respect to μ (or μ-ML-test) is a sequence $(U_n)_n$ of uniformly c.e. open sets with $\mu(U_n) \leq 2^{-n}$ for all $n \in \omega$. A sequence $X \in 2^\omega$ is called *μ-ML-random* if $X \notin \bigcap_n U_n$ for each μ-ML-test.

Let μ be a computable measure. A generalization of the Levin–Schnorr theorem states that the following conditions are equivalent:

(i) $\mathrm{KA}(X \upharpoonright n) > -\log \mu(X \upharpoonright n) - O(1)$.
(ii) $K(X \upharpoonright n) + \log \mu(X \upharpoonright n) \to \infty$ as $n \to \infty$.
(iii) $K(X \upharpoonright n) > -\log \mu(X \upharpoonright n) - O(1)$.

(iv) X is μ-Martin-Löf random.

Here, $X \upharpoonright n$ denotes the initial segment of X of length n. The equivalences among (i), (iii), and (iv) are straightforward, while the equivalence with (ii) follows from the ample excess lemma (see [7] for the uniform measure and [8, Corollary 2.32] for general computable measures).

2.2. Partial randomness

Although this paper does not primarily deal with partial randomness, we make use of several notions that originate from this area.

The study of *partial randomness* seeks to capture intermediate degrees of algorithmic randomness. Rather than the binary distinction between computable and Martin-Löf random sequences, it offers a quantitative scale of randomness, typically expressed in terms of the growth rate of Kolmogorov complexity. While partial randomness is closely related to effective Hausdorff dimension (see [9] for a survey), we confine our attention to partial randomness itself.

Terminology in the literature varies somewhat; in this paper we follow Hudelson's doctoral dissertation [3]. In particular, we focus on several key notions introduced there.

Later, we define $f(\sigma) = -\log \mu(\sigma)$ for a c.e. semimeasure μ. When $\mu(\sigma) = 0$, we set $f(\sigma) = +\infty$. Thus, f is a function from $2^{<\omega}$ to $[0, \infty]$. This extension of the codomain requires no change in the subsequent definitions or results.

Definition 2.1 (complexity; Chapter 4 in [3])**.** Let $f : 2^{<\omega} \to [0, \infty]$ be a function. A sequence $X \in 2^\omega$ is said to be

(I) KA-*f-complex* if $\mathrm{KA}(X \upharpoonright n) > f(X \upharpoonright n) - O(1)$;

(II) *strongly K-f-complex* if $K(X \upharpoonright n) - f(X \upharpoonright n) \to \infty$ as $n \to \infty$;

(III) *K-f-complex* if $K(X \upharpoonright n) > f(X \upharpoonright n) - O(1)$.

Definition 2.2 (test; Chapter 3 in [3])**.** Let $f : 2^{<\omega} \to [0, \infty]$ and $A \subseteq 2^{<\omega}$. The *direct f-weight* of A is

$$\mathrm{dwt}_f(A) = \sum_{\sigma \in A} 2^{-f(\sigma)}.$$

The *prefix-free f-weight* of A is the supremum of $\mathrm{dwt}_f(P)$ over all prefix-free subsets $P \subseteq A$:

$$\mathrm{pwt}_f(A) = \sup\{\mathrm{dwt}_f(P) \ : \ P \subseteq A \text{ is prefix-free } \}.$$

A *dwt-f-test* is a sequence $(A_n)_n$ of uniformly c.e. sets such that $\mathrm{dwt}_f(A_n) \leq 2^{-n}$ for all $n \in \omega$. A sequence $X \in 2^\omega$ is *dwt-f-random* if $X \notin \bigcap_n [A_n]$ for each dwt-f-test. A *pwt-f-test* and *pwt-f-randomness* are defined similarly.

A c.e. set $A \subseteq 2^{<\omega}$ is a *Solovay dwt-f-test* if $\mathrm{dwt}_f(A) < \infty$. A sequence $X \in 2^\omega$ is *Solovay dwt-f-random* if $A \cap \{\sigma \in 2^{<\omega} \ : \ \sigma \prec X\}$ is finite for any Solovay dwt-f-test A.

Theorem 2.3 (Theorem 4.1.8, 4.1.7, and 4.1.6 in [3]). *Let $f : 2^{<\omega} \to [0, \infty]$ be a computable function.*

- *A sequence $X \in 2^\omega$ is* KA*-f-complex if and only if X is pwt-f-random.*
- *A sequence $X \in 2^\omega$ is strongly K-f-complex if and only if X is Solovay dwt-f-random.*
- *A sequence $X \in 2^\omega$ is K-f-complex if and only if X is dwt-f-random.*

Theorem 2.4 (Theorem 3.3.9 and 3.3.7 in [3], Corollary 4.12 in [10]). *Let $f : 2^{<\omega} \to [0, \infty]$ be a computable function. If X is pwt-f-random, then X is Solovay dwt-f-random.*

Theorem 2.5 (Theorem 3.3.7 and 3.3.3 in [3]). *Let $f : 2^{<\omega} \to [0, \infty]$ be a computable function. If X is Solovay dwt-f-random, then X is dwt-f-random.*

The converses of these two implications do not hold in general as shown in Reimann and Stephan [11, Theorem 4.5, 4.7].

Historically, Tadaki [12] and Calude, Staiger, and Terwijn [13] investigated randomness with respect to $\mu(\sigma) = 2^{-s|\sigma|}$ where $s \in (0, 1]$. Later, Higuchi, Hudelson, Simpson, and Yokoyama [10] and Simpson [14] generalized the linear bound sn to an arbitrary computable function $f : 2^{<\omega} \to [0, \infty]$ and studied stability under relative partial randomness.

The results above are summarized in Figure 1, adapted from Figure 1.2 in [3].

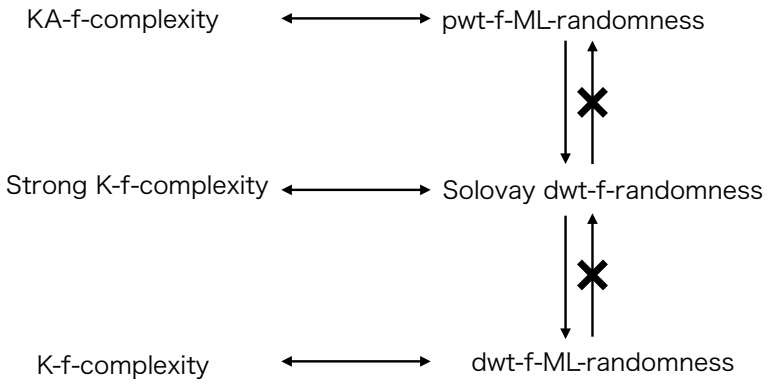

Figure 1: Implications for computable functions f from Hudelson [3]

2.3. Randomness with respect to c.e. semimeasures

We review notions of randomness with respect to c.e. semimeasures. Bienvenu et al. [1] were the first authors who studied randomness with respect to c.e. semimeasures. Their motivation was to determine whether randomness preservation and no-randomness-from-nothing results continue to hold for partial computable functions. Such functions naturally induce c.e. semimeasures.

By MLR, we denote the set of all ML-random sequences in 2^ω with respect to the uniform measure. To recall classical results, consider a Turing functional Φ defined almost everywhere. Then the measure induced by Φ is a computable measure.

Theorem 2.6 (Randomness preservation; [15, Theorem 4.2.b)])**.** *Let Φ be a Turing functional defined almost everywhere, and let μ denote the measure induced by Φ. If $X \in \mathrm{MLR}$, then $\Phi(X)$ is ML-random with respect to μ.*

Theorem 2.7 (No-randomness-from-nothing; [16, Theorem 5.1])**.** *Let Φ be a Turing functional defined almost everywhere, and let μ denote the measure induced by Φ. If Y is ML-random with respect to μ, then there exists a sequence $X \in \mathrm{MLR}$ such that $\Phi(X) = Y$.*

It is natural to ask whether these results can be generalized to arbitrary Turing functionals whose induced measures are c.e. semimeasures. Bienvenu et al. [1] defined Martin-Löf randomness with respect to a c.e. semimeasure μ as follows.

Definition 2.8 ((IV) ML-randomness for c.e. semimeasures; [1, Definition 5.1]). A *f-ML-test* is a sequence of uniformly c.e. sets $S_n \subseteq 2^{<\omega}$ such that $\sum_{\sigma \in S_n} 2^{-f(\sigma)} \leq 2^{-n}$ for all $n \in \omega$. A sequence $X \in 2^\omega$ is called *f-ML-random* if $X \notin \bigcap_n [S_n]$ for each f-ML-test $(S_n)_n$.

This notion coincides with dwt-f-randomness when $f = -\log \mu$. As we will see shortly, there is no need to distinguish among the three test-based notions of randomness defined in Definition 2.2. We simply refer to dwt-f-randomness as μ-ML-randomness. Observe that, if μ is a c.e. semimeasure, then $f = -\log \mu$ is an upper semicomputable function.

The principle of randomness preservation continues to hold for c.e. semimeasures.

Theorem 2.9 (Randomness preservation for c.e. semimeasures; [1, Theorem 5.4]). *Let Φ be a Turing functional and $X \in \mathrm{MLR} \cap \mathrm{dom}(\Phi)$. If μ is the measure induced by Φ, then $Y = \Phi(X)$ is μ-ML-random.*

More recently, Barmpalias and Shen [2, Proposition 4] established a no-randomness-from-nothing result for c.e. semimeasures. The following theorem restates their result in our terminology.

Theorem 2.10 (No-randomness-from-nothing for c.e. semimeasures). *Let Φ be a Turing functional that uses only the first $2n + o(1)$ inputs to produce n output bits. Let μ be the c.e. semimeasure induced by Φ and $f = -\log \mu$. If Y is* KA-*f-complex, then there exists a sequence $X \in \mathrm{MLR}$ such that $\Phi(X) = Y$.*

Barmpalias and Shen [2] also obtained a related result showing that the same conclusion holds under weaker assumptions on oracle use. Since the formal statement is rather lengthy and not central to the present paper, we omit it here.

2.4. Contributions

In this paper, we study the six notions introduced in Definitions 2.1 and 2.2 with respect to c.e. semimeasures. While these notions themselves are not new, previous studies have typically assumed that the underlying function f is computable. To the best of our knowledge, the case of non-computable f has not been systematically explored. Here we investigate randomness relative to an upper semicomputable function f, which is a weaker assumption.

In the study of partial randomness, it is common to restrict attention to convex functions f.

Definition 2.11 (convex; [3, Definition 3.4.14])**.** A function $f : 2^{<\omega} \to [0, \infty]$ is called *convex* if

$$\mathrm{dwt}_f(\sigma) \leq \mathrm{dwt}_f(\sigma 0) + \mathrm{dwt}_f(\sigma 1).$$

For example, let $\mu(\sigma) = 2^{-s|\sigma|}$ and $f(\sigma) = -\log \mu(\sigma) = s|\sigma|$ for some rational $s \in (0, 1)$. Then, f is a convex function because

$$\mathrm{dwt}_f(\sigma 0) + \mathrm{dwt}_f(\sigma 1) = 2^{-s(|\sigma|+1)} + 2^{-s(|\sigma|+1)} = 2^{-s|\sigma|+1-s} > 2^{-s|\sigma|} = \mathrm{dwt}_f(\sigma).$$

In contrast, we focus on the case where f is concave.

Definition 2.12 (concave)**.** A function $f : 2^{<\omega} \to [0, \infty]$ is called *concave* if

$$\mathrm{dwt}_f(\sigma) \geq \mathrm{dwt}_f(\sigma 0) + \mathrm{dwt}_f(\sigma 1).$$

The main topic of this paper is Martin-Löf randomness with respect to c.e. semimeasures. Equivalently, we study f-randomness and f-complexity for upper semicomputable and concave functions f.

We summarize our main contributions using the terminology introduced above.

(1) Tests for concave functions

When f is concave, the three test-based randomness notions defined in Definition 2.2 coincide. Recall that without the concavity assumption, this hierarchy does not collapse. As a corollary, for a computable and concave function f, the three test-based and the three complexity-based notions are all equivalent.

(2) Complexity for upper semicomputable and concave functions

Let f be a concave function that is upper semicomputable but not necessarily computable. Then, the three complexity-based notions in Definition 2.1 remain distinct. Since f may be non-computable, the correspondence between complexity and test formulations breaks down. Concavity collapses the test hierarchy but not the complexity hierarchy; in fact, K-f-complexity turns out to be a stronger notion than μ-ML-randomness.

Consequently, we obtain four distinct notions of randomness for c.e. semimeasures. The relationships among them are summarized in Figure 2.

(3) Randomness preservation and no-randomness-from-nothing

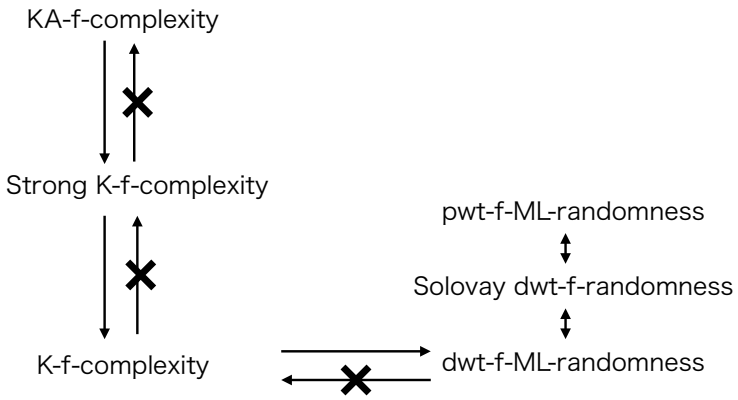

Figure 2: Implications for upper semicomputable and concave functions f

We show that the randomness preservation property holds for strong K-f-complexity (Theorem 6.1), strengthening the result of Bienvenu et al. [1] concerning μ-ML-randomness (Theorem 2.9). We further prove that, in general, randomness preservation does not hold for KA-f-complexity (Theorem 6.2), indicating that the assumption of strong K-f-complexity in the preservation result cannot be weakened to KA-f-complexity.

We also show that the no-randomness-from-nothing property fails for strong K-f-complexity, even when oracle use is restricted (Theorem 6.3). In other words, the condition on KA-f-complexity in the no-randomness-from-nothing result of Barmpalias and Shen [2] (Theorem 2.10) cannot be weakened to strong K-f-complexity, even under restricted oracle use.

3. Randomness by tests

In this section, we demonstrate that the hierarchy of randomness notions defined by tests collapses for c.e. semimeasures. The proof is straightforward, though its definition requires some care. However, we must discuss this, as the hierarchy of randomness notions defined by complexity does not collapse for c.e. semimeasures.

Although the relationship between c.e. open sets and prefix-free sets is well known in algorithmic randomness (see Downey and Hirschfeldt [5, Prop. 2.19.2] and Nies [4, Fact 1.8.26]), the following lemma gives a more explicit form of this correspondence. In particular, when dealing with randomness with respect to c.e. semimeasures, this formulation is convenient for

constructing prefix-free refinements of finite open sets. Although the result is implicit in the cited works, we restate and prove it here for completeness.

Lemma 3.1. *For a finite prefix-free set S and a string σ, one can compute a finite set T such that*

(i) *each string in T is an extension of σ,*
(ii) *$S \cup T$ is prefix-free, and*
(iii) *$[S] \cup [\sigma] = [S] \cup [T]$.*

Proof. If $S \cup \{\sigma\}$ is prefix-free, then let $T = \{\sigma\}$. If there exists $\tau \in S$ such that $\tau \preceq \sigma$, then let $T = \emptyset$. In the following, we restrict our attention to the remaining case and suppose that there exists $\tau \in S$ such that $\sigma \prec \tau$.

Let U be the set of all such strings:

$$U = \{\tau \in S \ : \ \sigma \prec \tau\}.$$

Let N be the maximum length of strings in U:

$$N = \max\{|\tau| \ : \ \tau \in U\}.$$

By the assumption, we have $N > |\sigma|$. Then, we define

$$T = \{\rho \ : \ \sigma \prec \rho, \ |\rho| = N \text{ and } U \cup \{\rho\} \text{ is prefix-free } \}.$$

Since any two pairs of strings in T have the same length, $U \cup T$ is prefix-free and so is $S \cup T$. If $|\rho| = N$, $\sigma \prec \rho$, and $\rho \notin T$, then we have $S \cup \{\rho\}$ is not prefix-free, which implies that $\tau \prec \rho$ for some $\tau \in U$ and $[\rho] \subseteq [S]$. Hence, we have $[S] \cup [\sigma] = [S] \cup [T]$. □

Proposition 3.2. *Let $f : 2^{<\omega} \to [0, \infty]$ be a function. Then, pwt-f-randomness implies dwt-f-randomness. If f is, in addition, concave, the converse is also true.*

Remark 3.3. We make no assumptions regarding the computability of f. Thus, $\mathrm{dwt}_f(\sigma)$ for $\sigma \in 2^{<\omega}$ may not be computable in general.

Proof. Suppose that $X \in 2^\omega$ is not dwt-f-random. Then, there exists a sequence $(A_n)_n$ of uniformly c.e. sets such that $\mathrm{dwt}_f(A_n) \leq 2^{-n}$ and $X \in [A_n]$ for all $n \in \omega$. By definition, $\mathrm{pwt}_f(A_n) \leq \mathrm{dwt}_f(A_n) \leq 2^{-n}$ for all $n \in \omega$. Thus, $(A_n)_n$ is a pwt-f-test. Hence, X is not pwt-f-random.

Suppose further that f is concave. Suppose that X is not pwt-f-random. Then, there exists a sequence $(A_n)_n$ of uniformly c.e. sets such that $\mathrm{pwt}_f(A_n) \leq 2^{-n}$ and $X \in [A_n]$ for all $n \in \omega$.

Fix $n \in \omega$. Let a computable enumeration of $A_n = \{\sigma_0, \sigma_1, \cdots\}$ be given, which may be finite. By Lemma 3.1, by induction, we can computably construct a sequence $(T_k)_k$ of finite sets such that

(i) each string in T_k is an extension of σ_k,
(ii) $\bigcup_k T_k$ is prefix-free, and
(iii) $[A_n] = \bigcup_k [T_k]$.

Let $S \subseteq A_n$ be the set of all strings σ such that each proper prefix of σ is not in A_n. Since S is a prefix-free subset of A_n, we have $\mathrm{dwt}_f(S) \leq 2^{-n}$. Each string in T_k is an extension of a string in A_n, and thus an extension of a string in S. Since f is concave, we have $\mathrm{dwt}_f(\bigcup_k T_k) \leq \mathrm{dwt}_f(S)$. Note that the set $\bigcup_k T_k$ is a c.e. set of strings while we do not impose any computability on S.

We use this construction for each n. Then, we can construct a sequence $(B_n)_n$ of uniformly c.e. sets such that $\mathrm{dwt}_f(B_n) \leq 2^{-n}$ and $X \in [B_n]$ for all $n \in \omega$. Thus, X is not dwt-f-random. □

If we further assume that f is computable, then by Theorems 2.3, 2.4, and 2.5, we obtain the following.

Corollary 3.4. *Let $f : 2^{<\omega} \to [0, \infty]$ be a computable concave function. Then, statements (I) through (IV) defined in Definition 2.1 and Definition 2.8 are equivalent.*

4. Logical implications among conditions (I)–(IV)

From now on, we do not assume that the function f is computable. We sometimes assume that f is upper semicomputable and sometimes do not assume any computability on f. Notice that condition (II) strong K-f-complexity implies (III) K-f-complexity obviously.

4.1. (I) implies (II)

We show that (I) KA-f-complexity implies (II) strong K-f-complexity. Andreev and Kumok [17] attribute Theorem 4.1 to an unpublished result by Lempp, Miller, Ng, and Turetsky (2010). While no proof is provided in their work, we establish the following result in this paper.

Theorem 4.1. *Let $X \in 2^\omega$. Then,*

$$K(X \upharpoonright n) - \mathrm{KA}(X \upharpoonright n) \to \infty \text{ as } n \to \infty.$$

Corollary 4.2. *Let $f : 2^{<\omega} \to [0, \infty]$ be a function. Then,* KA-f*-complexity implies strong* K*-*f*-complexity.*

Notice that we do not assume any computability on f in the above claim.

Remark 4.3. This implication has been shown for a computable function f through the equivalence with strong f-ML-randomness and Solovay f-randomness, respectively; see Hudelson [3].

We will establish this theorem by adapting the proof of [10, Theorem 4.5].

Proof. Let μ be the uniform measure on 2^ω and MLR be the set of all Martin-Löf random sequences with respect to μ. Given $X \in 2^\omega$, let $Y \in 2^\omega$ be such that $X \leq_T Y \in \mathrm{MLR}$ by the Kučera-Gács theorem. Let Φ be a Turing reduction such that $X = \Phi^Y$.

For each $\sigma \in 2^{<\omega}$, we define V_σ by

$$V_\sigma = \{\overline{Y} \ : \ \sigma \preceq \Phi^{\overline{Y}}\}.$$

Notice that, by definition, we have $Y \in V_{X \upharpoonright n}$ for all n. We define ν by $\nu(\sigma) = \mu(V_\sigma)$. Then, $\nu(\sigma)$ is a c.e. semimeasure. Let ξ be an optimal c.e. semimeasure such that $\mathrm{KA} = -\log \xi$. Then, there exists $c_0 \in \omega$ such that $\mu(V_\sigma) = \nu(\sigma) \leq c_0 \xi(\sigma)$ for all $\sigma \in 2^{<\omega}$.

Suppose there exists $c_1 \in \omega$ such that $K(X \upharpoonright n) < \mathrm{KA}(X \upharpoonright n) + c_1$ for infinitely many $n \in \omega$. Then,

$$\mu(V_{X \upharpoonright n}) < c_0 2^{c_1 - K(X \upharpoonright n)}$$

for infinitely many n.

Let U be the universal prefix-free machine to define K. Consider the set

$$W = \bigcup_{\tau \in \mathrm{dom}(U)} \widehat{V_{U(\tau)}}$$

where $\widehat{V_{U(\tau)}}$ is $V_{U(\tau)}$ enumerated as long as $\mu(V_{U(\tau)}) < c_0 2^{c_1 - |\tau|}$. The weight of the set W is bounded by

$$\sum_{\tau \in \mathrm{dom}(U)} \mu(\widehat{V_{U(\tau)}}) \leq \sum_{\tau \in \mathrm{dom}(U)} c_0 2^{c_1 - |\tau|} < \infty.$$

Thus, W is a Solovay test, as its total weight is bounded.

We claim that $Y \in [\tau]$ for infinitely many $\tau \in W$. By the assumption, there are infinitely many $\sigma \prec X$ such that $Y \in V_\sigma,\ \mu(V_\sigma) < c_0 2^{c_1 - K(\sigma)}$. Thus, there are infinitely many $\tau \in \mathrm{dom}(U)$ such that $Y \in \widehat{V_{U(\tau)}} = V_{U(\tau)}$.

This argument implies that Y is not ML-random, a contradiction. □

Remark 4.4. We note that

$$\mathrm{KA}(0^n 1) = K(0^n) + O(1).$$

This fact can be shown analogously to its monotone complexity version [5, Proposition 3.15.3]. Thus, a stronger version of the theorem, $\mathrm{KA}(\sigma) - K(\sigma) \to \infty$ as $|\sigma| \to \infty$, does not hold.

4.2. (III) implies (IV)

We show that (III) K-f-complexity implies (IV) f-ML-randomness. Hudelson [3, Theorem 4.1.6] has shown the equivalence between (III) and (IV) for a computable function f. The implication from (III) to (IV) can also be shown for an upper semicomputable f, more or less using the same proof. A counterexample to the converse will be given in a later section.

Proposition 4.5. *Let $f : 2^{<\omega} \to [0, \infty]$ be an upper semicomputable function. Then, K-f-complexity implies f-ML-randomness.*

Proof. Let $(A_n)_n$ be a sequence of uniformly c.e. sets such that $\mathrm{dwt}_f(A_n) \leq 2^{-n}$ and $X \in [A_n]$ for all $n \in \omega$. To use the KC theorem, we construct a KC-set S as follows. For each n, each $\sigma \in A_{2n}$, and each $k \in \omega$ such that $k > f(\sigma) - n$, we computably enumerate (σ, k) into S. Since A_n is c.e. and f is upper semicomputable, this is possible. Then, the weight of S is

$$\begin{aligned}\sum_{(\sigma,k)} 2^{-k} &= \sum_n \sum_{\sigma \in A_{2n}} \sum_{k > f(\sigma) - n} 2^{-k} < \sum_n \sum_{\sigma \in A_{2n}} 2^{-f(\sigma)+n+1} \\ &= \sum_n 2^{n+1} \mathrm{dwt}_f(A_{2n}) \leq \sum_n 2^{-n+1} < \infty.\end{aligned}$$

By the KC theorem, we have $K(\sigma) \leq f(\sigma) - n + O(1)$ for all $\sigma \in A_{2n}$. Since $X \in [A_n]$ for all $n \in \omega$, X is not K-f-complex. □

The proof of the converse for an upper semicomputable function f fails. This is because $\{\sigma \ :\ K(\sigma) < f(\sigma) - n\}$ need not be a c.e. set.

5. Counterexamples to the reverse implications

5.1. Counterexample to the implication from (IV) to (III)

In this section, we show that the reverse implication from (IV) f-ML-randomness to (III) K-f-complexity does not hold.

Theorem 5.1. *There exist a c.e. semimeasure ν and a sequence $X \in 2^\omega$ such that X is f-ML-random but not K-f-complex where $f = -\log \nu$.*

Proof. We construct such a c.e. semimeasure ν and a sequence X using a finite-injury argument.

Let $(U^e_k)_k$ be the e-th sequence of c.e. sets of $2^{<\omega}$. To ensure that X is f-ML-random, if $\nu(U^e_k) \leq 2^{-k}$ for all $k \in \omega$, we need to ensure $X \notin \bigcap_k U^e_k$. We set the requirement

$$Q_e : \ (\exists k) X \in U^e_k \implies \nu(U^e_k) > 2^{-k}.$$

If $(U^e_k)_k$ is a ν-ML-random test and the requirement Q_e is met, then $X \notin U^e_k$ for the witness of Q_e and X passes the test $(U^e_k)_k$. Thus, if all requirements Q_e are met, then X is ν-ML-random.

We also set the requirement

$$R_e : \ (\exists n) K(X \upharpoonright n) < -\log \nu(X \upharpoonright n) - e + c,$$

where $c \in \omega$ will be defined later. If all requirements R_e are met, then X is not K-f-complex.

We set their priorities as follows:

$$Q_0 > R_0 > Q_1 > R_1 > \cdots .$$

Each requirement forces a corresponding string as follows:

$$\sigma_0 \prec \tau_0 \prec \sigma_1 \prec \tau_1 \cdots \prec X.$$

We define τ_{-1} as the empty string for convenience.

The strategy to satisfy Q_e is as follows. Choose a sufficiently large $k \in \omega$. Then, consider the following Π^0_1-relation: $[\tau_{e-1}] \cap U^e_k = \emptyset$. If this is true, we define σ_e as an extension of τ_{e-1}. Otherwise, pick a string $\sigma_e \in 2^{<\omega}$ such that $\tau_{e-1} \prec \sigma_e$, $[\sigma_e] \subseteq U^e_k$ with some additional conditions. Then we define $\nu(\sigma_e)$ so that $\nu(\sigma_e) > 2^{-k}$. Since the relation is not decidable, the requirement Q_e may cause an injury.

The strategy to satisfy R_e is as follows. Pick $\tau_e \succ \sigma_e$ such that $K(\tau_e)$ is sufficiently small and ensure $\nu(\tau_e)$ remains small. This strategy does not need an injury.

We can modify σ_e, τ_e, as long as it is done in a computable manner. We denote σ_e, τ_e at stage s by $\sigma_{e,s}, \tau_{e,s}$. Thus, the sequence $X \in 2^\omega$ will be in Δ^0_2.

We must ensure the function ν is a c.e. semimeasure. Therefore, we can only approximate $\nu(\rho)$ from below. We denote the approximation of $\nu(\rho)$ at stage s by $\nu_s(\rho)$. Let

$$W_s = \{\sigma_{e,t} \ : \ t \leq s\} \cup \{\tau_{e,t} \ : \ t \leq s\}.$$

At each stage s, we only consider ν on W_s as ν can be extended to all strings in a computable manner. If $\rho \in 2^{<\omega}$ is not a prefix of each string in W_s, then $\nu_s(\rho)$ at stage s is 0. To ensure that we can always select such a string, we impose the following conditions at each stage s:

$$[\tau_{e-1}] \subsetneq \bigcup_\rho \{[\rho] \ : \ \tau_{e-1} \prec \rho \in W_s\}, \tag{1}$$

$$[\sigma_e] \subsetneq \bigcup_\rho \{[\rho] \ : \ \sigma_e \prec \rho \in W_s\} \tag{2}$$

Construction.

We define strings $\sigma_0, \tau_0, \sigma_1, \tau_1, \cdots$ one by one. Each requirement Q_e or R_e is associated with k, n respectively.

At stage s, assuming that no injury occurs, pick the smallest index e such that σ_e is undefined. We define σ_e, τ_e at stage s (or $\sigma_{e,s}, \tau_{e,s}$) and σ_i, τ_i for $i > e$ continue to be undefined.

The construction of σ_e is as follows. Pick $k \in \omega$ such that

$$2^{-s-2}\nu_s(\tau_{e-1}) > 2^{-k}.$$

Since $k > s$, $U^e_{k,s}$ is the empty set by the usual convention where $U^e_{k,s}$ is the clopen approximation of U^e_k at stage s. Then, pick any string σ_e such that

- $\tau_{e-1} \prec \sigma_e$,
- σ_e is neither a prefix of any string in W_{s-1} nor an element of W_{s-1}.

We also impose the condition (1) by extending if necessary. We also define

$$\nu_s(\sigma_e) = 2^{-s-2}\nu_{s-1}(\tau_{e-1}) \tag{3}$$

and $\nu_s(\rho) = \nu_{s-1}(\rho)$ for each $\rho \in W_{s-1}$. In this case, Q_e is associated with k.

The construction of τ_e is as follows. Pick $\tau_e \in 2^{<\omega}$ such that

- $\sigma_e \prec \tau_e$,
- τ_e is neither a prefix of any string in W_{s-1} nor an element in W_{s-1}.

We also impose the condition (2) by extending if necessary. Then, we define

$$\nu_s(\tau_e) = \min\{2^{-s-1}\nu(\sigma_e), 2^{-2|\tau|-e}\}$$

and $\nu_s(\rho) = \nu_{s-1}(\rho)$ for each $\rho \in W_{s-1}$. In this case, R_e is associated with $n = |\tau_e|$.

Suppose that σ_e is defined and Q_e is associated with k. The requirement Q_e requires attention at stage s if $[\tau_{e-1}] \cap U^e_{k,s} \neq \emptyset$. If so, we say that Q_e causes an injury.

Let Q_e be the requirement requiring attention at stage s with the highest priority. We define σ_e, τ_e at stage s and reset σ_i, τ_i for $i > e$ to be undefined.

In this case, the construction of $\sigma_e = \sigma_{e,s}$ is as follows. Among the strings

$$\sigma_{e,s-1},\ \tau_{e,s-1},\ \sigma_{e+1,s-1},\ \tau_{e+1,s-1},\ \cdots$$

let ρ be the shortest one defined and satisfying $[\rho] \subseteq U^e_{k,s}$. Then, define $\sigma_e \in 2^{<\omega}$ such that

- $\rho \prec \sigma_e$,
- $[\sigma_e] \subseteq U^e_{k,s}$.
- σ_e is neither a prefix of any string in W_{s-1} nor an element in W_{s-1}.

Again, we impose the condition (1). We define

$$\nu_s(\sigma_e) = \nu_{s-1}(\sigma_{e,s-1}). \tag{4}$$

The construction of τ_e is the same as above.

Finally, let $X \in 2^\omega$ be the unique one such that $\sigma_e, \tau_e \prec X$ for all e. This is the end of the construction.

Verification.

We claim that this construction is possible. In particular, one can find such σ_e, τ_e at each stage s. This is because of condition (1) and (2) and by induction.

Each requirement requires attention at most once after all requirements with higher priorities settle their strings. Thus, by induction, all strings σ_e, τ_e will be settled eventually.

For each index e, if the requirement Q_e does not require attention after all requirements with higher priority settle down, then $X \notin U^e_k$ where k is the associated integer of Q_e, and hence Q_e is met. If the requirement Q_e requires attention, then

$$\nu(U^e_k) \geq \nu(\sigma_e) = 2^{-s-2}\nu(\tau_{e-1})[s] > 2^{-k},$$

where s is the stage at which k is defined. Thus, Q_e is met.

For the requirement R_e,

$$K(\tau_e) \leq 2|\tau_e| + c \leq -\log \nu(\tau_e) - e + c,$$

and $\tau_e \prec X$. Here, we define $c \in \omega$ so that the inequality above holds for all strings. Thus, R_e is met.

We show that ν is a c.e. semimeasure. When updating ν at stage s, we always define a new string that is not a prefix of any string in W_{s-1}. Thus, ν increases from 0.

For each $\tau_{e-1,s}$, by equation (3) and (4), we have

$$\sum_{\rho} \{\nu(\rho) \ : \ \tau_{e-1} \prec \rho \in W_s\} \leq \nu(\tau_{e-1}).$$

The corresponding inequality also holds for each $\sigma_{e,s}$. Thus, ν can be computably extended as a semimeasure. □

5.2. Counterexample to the Implication from (III) to (II)

In this section, we show that the reverse implication from (III) K-f-complexity to (II) strong K-f-complexity does not hold.

Theorem 5.2. *There exists a c.e. semimeasure ν such that the sequence 0^ω is K-f-complex but not strongly K-f-complex, where $f = -\log \nu$.*

Here, 0^ω refers to the infinite binary sequence consisting of all zeros. In the below, 0^n denotes a finite binary string consisting of n zeros.

Proof. We define ν as follows. For any string σ that is not of the form 0^n, set $\nu(\sigma) = 0$. For strings of the form 0^n, define $\nu(0^n)$ by

$$-\log \nu(0^n) = \min_{m \geq n} K(0^m).$$

Since the right-hand side is non-decreasing in n, ν is a semimeasure. Moreover, because the right-hand side is upper semicomputable, ν is lower semicomputable.

Since $\min_{m \geq n} K(0^m) \leq K(0^n)$ for all n, the sequence 0^ω is K-f-complex. However, because $\min_{m \geq n} K(0^m) = K(0^n)$ for infinitely many n, 0^ω is not strongly K-f-complex. □

5.3. Counterexample to the Implication from (II) to (I)

In this section, we show that the reverse implication from (II) strong K-f-complexity to (I) KA-f-complexity does not hold.

Theorem 5.3. *There exists a c.e. semimeasure ν such that the sequence 0^ω is strongly K-f-complex but not* KA-f-*complex, where $f = -\log \nu$.*

Proof. The construction of ν is similar to that used in Theorem 5.2. For any string σ that is not of the form 0^n, set $\nu(\sigma) = 0$. For strings of the form 0^n, define $\nu(0^n)$ by

$$-\log \nu(0^n) = \frac{1}{2} \min_{m \geq n} K(0^m).$$

Then, ν is a c.e. semimeasure.

Since $\mathrm{KA}(0^n) = O(1)$ and $K(0^n) \to \infty$, $f(0^n) \to \infty$ and the sequence 0^ω is not KA-f-complex. We also have

$$K(0^n) - f(0^n) \geq \frac{1}{2} \min_{m \geq n} K(0^m) \to \infty,$$

which means that the sequence 0^ω is strongly K-f-complex. □

6. Randomness preservation and no-randomness-from-nothing principles

6.1. Randomness preservation

By MLR, we denote the set of all ML-random sequences in 2^ω with respect to the uniform measure. Bienvenu et al. [1, Theorem 5.4] showed that

the randomness preservation property holds for μ-ML-randomness (Theorem 2.9). Thus, the image of ML-random sequences is contained in ML-randomness w.r.t. the induced measure. We show that this result can be extended to (II) strong K-f-complexity but not (I) KA-f-complexity.

Theorem 6.1. *Let Φ be a Turing functional and $Y \in \mathrm{MLR} \cap \mathrm{dom}(\Phi)$. Then, $X = \Phi(Y)$ is strongly K-f-complex, where ν is the induced measure of Φ from the uniform measure μ and $f = -\log \nu$.*

Building upon the proof technique used in Theorem 4.1, we give a proof of this result.

Proof. We define $V_\sigma = \{\overline{Y} \ : \ \sigma \preceq \Phi^{\overline{Y}}\}$. Suppose there exists $c_1 \in \omega$ such that $K(X \upharpoonright n) < f(X \upharpoonright n) + c_1$ for infinitely many $n \in \omega$. Then,

$$\mu(V_{X \upharpoonright n}) = \nu(X \upharpoonright n) < 2^{c_1 - K(X \upharpoonright n)}$$

for infinitely many n.

Let U be the universal prefix-free machine to define K. Consider the set

$$W = \bigcup_{\tau \in \mathrm{dom}(U)} \widehat{V_{U(\tau)}}$$

where $\widehat{V_{U(\tau)}}$ is $V_{U(\tau)}$ enumerated as long as $\mu(V_{U(\tau)}) < 2^{c_1 - |\tau|}$. The weight of the set W is bounded by

$$\sum_{\tau \in \mathrm{dom}(U)} \mu(\widehat{V_{U(\tau)}}) \leq \sum_{\tau \in \mathrm{dom}(U)} 2^{c_1 - |\tau|} < \infty.$$

Thus, W is a Solovay test.

We claim that $Y \in [\tau]$ for infinitely many $\tau \in W$. By the assumption, there are infinitely many $\sigma \prec X$ such that $Y \in V_\sigma,\ \mu(V_\sigma) < 2^{c_1 - K(\sigma)}$. Thus, there are infinitely many $\tau \in \mathrm{dom}(U)$ such that $Y \in \widehat{V_{U(\tau)}} = V_{U(\tau)}$.

This argument implies that Y is not ML-random, a contradiction. □

Theorem 6.2. *There exists a Turing functional $\Phi :\subseteq 2^\omega \to 2^\omega$ and an ML-random sequence $X \in 2^\omega$ such that $Y = \Phi(X) \downarrow$ is not* KA-f-*complex, where ν is the induced measure of Φ from the uniform measure λ and $f = -\log \nu$.*

Proof. Let Φ be the Turing functional defined in [1, Theorem 4.1]. Then, $\mathrm{dom}(\Phi) = \{\Omega\}$, $\Phi(\Omega) = 0^\omega$. Furthermore, $\nu(0^n) = 2^{-n} \cdot t_n$ where t_n is the number of possible $\Omega_s \upharpoonright n$. Figueira et al. [18, Fact 2] showed that $t_n = o(2^n)$. Thus, $f(0^n) = -\log \nu(0^n) \to \infty$ as $n \to \infty$. Since 0^ω is computable, $\mathrm{KA}(0^n) = O(1)$. Hence, the sequence 0^ω is not KA-f-complex. □

The example in the proof is interesting and we will look at it in more detail.

Let Ω be a left-c.e. ML-random real and fix a computable approximation Ω_s from below. We define a Turing functional Φ by $(\Omega_s \upharpoonright n, 0^n) \in \Phi$ for each s and n. Then, $\mathrm{dom}(\Phi) = \{\Omega\}$, $\Phi(\Omega) = 0^\omega$. Let ν be the induced measure of Φ from the uniform measure λ and let $f = -\log \nu$.

We show that no sequences are KA-f-complex. Since the number t_n of $\Omega_s \upharpoonright n$ is $o(2^n)$, $\nu(0^n) \to 0$ and $f(0^n) \to \infty$ as $n \to \infty$. Thus, 0^ω is not a KA-f-complex sequence. Let $X \in 2^\omega$ be such that $X \neq 0^\omega$. Then, there exists $n \in \omega$ such that $\nu(X \upharpoonright m) = 0$ and $f(X \upharpoonright m) = \infty$ for each $m \geq n$. Hence, X is not KA-f-complex.

The sequence 0^ω is the unique sequence that is strongly K-f-complex. Note that 0^ω is strongly K-f-complex by randomness preservation. Thus, the convergence speed of t_n is very slow in the sense that

$$K(n) - \log t_n \to \infty \ (n \to \infty).$$

Other sequences are not strongly K-f-complex for the same reason as above.

6.2. No-randomness-from-nothing

The No-randomness-from-nothing principle for a c.e. semimeasure does not generally hold, as shown in Bienvenu et al. [1, Theorem 4.1]. Barmpalias and Shen [2, Proposition 4] recently showed no-randomness-from-nothing for (I) KA-f-complex sequences with oracle use bound $2n+O(1)$ (Theorem 2.10). Even with stricter bound of oracle use, this result cannot be extended to (II) strong K-f-complexity.

Theorem 6.3. *There exists a Turing functional Φ with use bound $n + O(1)$ and a sequence $Y \in 2^\omega$ such that*

- *Y is strongly K-f-complex,*
- *$Y \neq \Phi(X)$ for each $X \in \mathrm{MLR} \cap \mathrm{dom}(\Phi)$,*

where ν is the induced measure of Φ from the uniform measure and $f = -\log \nu$.

The proof is based on Theorem 5.3 and Bienvenu et al. [1, Theorem 4.1].

Proof. We use the c.e. semimeasure ν as Theorem 5.3. For any string σ that is not of the form 0^n, set $\nu(\sigma) = 0$. For strings of the form 0^n, define $\nu(0^n)$ by

$$-\log \nu(0^n) = \frac{1}{2} \min_{m \geq n} K(0^m).$$

We construct a Turing functional Φ such that ν is the induced measure of Φ as follows: For each $n \in \omega$, we list all pairs $(\sigma, 0^n) \in \Phi$ where σ is a binary string of length n, in lexicographic order. The total number of such pairs is $2^{n-f(0^n)}$. Note that ν is the induced measure of Φ.

As shown in Theorem 5.3, $Y = 0^\omega$ is strongly K-f-complex. Since $f(0^n) \to \infty$ as $n \to \infty$, we have $\mathrm{dom}(\Phi) = \{0^\omega\}$ by the same argument of Bienvenu et al. [1, Theorem 4.1]. Hence, Y has no ML-random sequence in the preimage of Φ. □

Acknowledgement

This research began during Laurent Bienvenu's one-year sabbatical at our institution in Tokyo, Japan, and the author is grateful to him for introducing this problem.

This work was supported by JSPS KAKENHI Grant Numbers JP22K03408 and JP25K07105. This work was also supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

References

[1] L. Bienvenu, R. Hölzl, C. P. Porter, P. Shafer, Randomness and semimeasures, Notre Dame Journal of Formal Logic 58 (3) (2017) 301–328. doi:10.1215/00294527-3839446.

[2] G. Barmpalias, A. Shen, The Kučera–Gács theorem revisited by Levin, Theoretical Computer Science 947 (2023) 113693. doi:https://doi.org/10.1016/j.tcs.2023.113693.

[3] W. P. Hudelson, Partial Randomness and Kolmogorov Complexity, PhD dissertation, The Pennsylvania State University, University Park, PA, submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy (May 2013).

[4] A. Nies, Computability and randomness, Vol. 51, Oxford University Press, 2009.

[5] R. G. Downey, D. R. Hirschfeldt, Algorithmic Randomness and Complexity, Theory and Applications of Computability, Springer, New York, 2010. doi:10.1007/978-0-387-68441-3.

[6] A. Shen, V. A. Uspensky, N. Vereshchagin, Kolmogorov Complexity and Algorithmic Randomness, American Mathematical Society, 2017.

[7] J. S. Miller, L. Yu, On initial segment complexity and degrees of randomness, Transactions of the American Mathematical Society 360 (2008) 3193–3210.

[8] L. Bienvenu, P. Gács, M. Hoyrup, C. Rojas, A. Shen, Algorithmic tests and randomness with respect to a class of measures, in: Proceedings of the Steklov Institute of Mathematics, Vol. 274, Springer, 2011, pp. 41–102.

[9] J. Reimann, Information vs. Dimension: An Algorithmic Perspective, World Scientific, 2020, Ch. 4, pp. 111–151.

[10] K. Higuchi, W. M. P. Hudelson, S. G. Simpson, K. Yokoyama, Propagation of partial randomness, Annals of Pure and Applied Logic 165 (2) (2014) 742–758. doi:10.1016/j.apal.2013.10.005.

[11] J. Reimann, F. Stephan, Hierarchies of randomness tests, in: Proceedings of the 9th Asian Logic Conference, World Scientific Publishing, Novosibirsk, Russia, 2006, pp. 215–232.

[12] K. Tadaki, A generalization of Chaitin's halting probability Ω and halting self-similar sets, Hokkaido Mathematical Journal 31 (1) (2002) 219–253. doi:10.14492/hokmj/1350911778.

[13] C. S. Calude, L. Staiger, S. A. Terwijn, On partial randomness, Annals of Pure and Applied Logic 138 (1-3) (2006) 20–30. doi:10.1016/j.apal.2005.07.004.

[14] S. G. Simpson, F. Stephan, Cone avoidance and randomness preservation, Annals of Pure and Applied Logic 166 (6) (2015) 713–728. doi:10.1016/j.apal.2015.03.001.

[15] A. K. Zvonkin, L. A. Levin, The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms, Russian Mathematical Surveys 25 (6) (1970) 83–124.

[16] L. Bienvenu, A. Romashchenko, A. Shen, Sparse sets, in: 1st Symposium on Cellular Automata "Journées Automates Cellulaires" (JAC 2008), MCCME Publishing House, Moscow, 2008, pp. 18–28.

[17] M. Andreev, A. Kumok, The Sum $2^{KM(x)-K(x)}$ Over All Prefixes x of Some Binary Sequence Can be Infinite, Theory Comput. Syst. 58 (3) (2016) 424–440. doi:10.1007/S00224-014-9604-2.

[18] S. Figueira, D. R. Hirschfeldt, J. S. Miller, K. M. Ng, A. Nies, Counting the changes of random Δ^0_2 sets, J. Log. Comput. 25 (4) (2015) 1073–1089. doi:10.1093/LOGCOM/EXS083.